Classification of Direct Limits of Even Cuntz-Circle Algebras

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Contents

Abstract

We prove a classification theorem for purely infinite simple C^*-algebras that is strong enough to show that the tensor products of two different irrational rotation algebras with the same even Cuntz algebra are isomorphic. In more detail, let $\mathcal{C}$ be the class of simple C^*-algebras A which are direct limits $A \cong \varinjlim A_k$, in which each A_k is a finite direct sum of algebras of the form $C(X) \otimes M_n \otimes \mathcal{O}_m$, where m is even, $\mathcal{O}_m$ is the Cuntz algebra, and X is either a point, a compact interval, or the circle S^1, and each map $A_k \to A$ is approximately absorbing. ("Approximately absorbing" is defined in Section 1.) We show that two unital C^*-algebras A and B in $\mathcal{C}$ are isomorphic if and only if

$$(K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B)).$$

This class is large enough to exhaust all possible K-groups: if G_0 and G_1 are countable odd torsion (abelian) groups and $g \in G_0$, then there is a C^*-algebra A in $\mathcal{C}$ with $(K_0(A), [1_A], K_1(A)) \cong (G_0, g, G_1)$. The class $\mathcal{C}$ contains the tensor products of irrational rotation algebras with even Cuntz algebras. It is also closed under the formation of hereditary subalgebras, countable direct limits (provided that the direct limit is simple), and tensor products with simple AF algebras.

Key Words: Even Cuntz-circle algebras, Classification of simple C^*-algebras, K-theory, Direct limits.

0 Introduction

We prove a classification theorem for simple direct limits of what we call even Cuntz-circle algebras: finite direct sums of algebras of the form $C(X) \otimes M_n \otimes \mathcal{O}_m$, where m is even, $\mathcal{O}_m$ is the Cuntz algebra (first introduced in [Cu1]), and X is either a point, a compact interval, or the circle S^1. The unital version of our main theorem is:

Theorem A. (Theorem 5.4) Let $A = \varinjlim A_k$ and $B = \varinjlim B_k$ be simple separable unital C^*-algebras, which are direct limits of even Cuntz-circle algebras. Assume that the homomorphisms $A_k \to A$ and $B_k \to B$ are "approximately absorbing" (defined in Section 1). Then $A \cong B$ if and only if $(K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B))$. In particular, if there are isomorphisms $\alpha_0 : K_0(A) \to K_0(B)$ and $\alpha_1 : K_1(A) \to K_1(B)$ such that $\alpha_0([1_A]) = [1_B]$, then A is isomorphic to B.

As a corollary, we obtain:

Theorem B. (Corollary 5.12) Let θ_1 and θ_2 be irrational numbers, and let A_{θ_1} and A_{θ_2} be the corresponding irrational rotation algebras. Then for any even m, we have $A_{\theta_1} \otimes \mathcal{O}_m \cong A_{\theta_2} \otimes \mathcal{O}_m$.

This contrasts with the fact, due to Rieffel [Rf] and Pimsner and Voiculescu [PV1] that $A_{\theta_1} \cong A_{\theta_2}$ only when $\theta_1 = \pm\theta_2 \pmod{\mathbf{Z}}$. (Theorem B was already known for $m = 2$ [Ln3], and remains unknown for odd m.) generally, tensor products of simple direct limits of circle or interval algebras (whether of real rank 0 or 1) with even Cuntz algebras are classified up to isomorphism by their K-theory. Further results along these lines are given in Section 5.

We do not know if maps from even Cuntz-circle algebras to simple direct limits of even Cuntz-circle algebras are necessarily approximately absorbing. Therefore, we do not classify arbitrary simple direct limits of even Cuntz-circle algebras. However, there are more general situations than Theorem B in which the approximately absorbing condition is automatic. For example, in Section 5 we will define "co-Cuntz algebras" $\mathcal{Q}_m$, which are unital purely infinite simple C^*-algebras satisfying $K_0(\mathcal{Q}_m) = 0$ and $K_1(\mathcal{Q}_m) \cong \mathbf{Z}/(m-1)\mathbf{Z}$. We prove, without any hypotheses on the maps of the systems:

Theorem C. (Theorem 5.24 (1)) Let $A = \varinjlim A_k$ and $B = \varinjlim B_k$ be simple unital direct limits, in which the A_k and B_k are each finite direct sums of matrix algebras over even Cuntz algebras and even co-Cuntz algebras. Then $A \cong B$ if and only if $(K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B))$.

We also prove that our class is closed under the formation of direct limits.

All feasible values of the invariant are actually realized:

Theorem D. (Theorem 5.26 (1)) Let G_0 and G_1 be countable odd torsion groups, and let $g_0 \in G_0$. Then there exists a unital C^*-algebra, in the classes covered by Theorems A and C, such that $(K_0(A), [1_A], K_1(A)) \cong (G_0, g_0, G_1)$.

Theorem A probably remains true if the condition on X is relaxed slightly, to allow arbitrary compact subsets of S^1 in the definition of a Cuntz-circle algebra. Proving this would make an already long paper even longer, and we would get no new values of the invariant. So we don't do it.

Our results are part of the general classification program for simple separable nuclear C^*-algebras. This program was initiated by George Elliott [Ell1] many years ago (1976) with the classification of AF algebras. Starting much more recently (about 1990), it has been extended to the class of C^*-algebras of real rank

zero which are direct limits of circle algebras by Elliott [Ell2], and to much larger classes of (simple) stably finite direct limit algebras by Elliott [Ell3], [Ell4], Su [Su1], [Su2], [Su3], Elliott and Gong [EG1], [EG2], Elliott, Gong, Lin and Pasnicu [EGLP], etc., and to classes of purely infinite simple C^*-algebras by Bratteli, Kishimoto, Rørdam and Størmer [BKRS] and by Rørdam [Rr1], [Rr2], [Rr3]. In addition to the classification of direct limits, one of the striking successes of this program is the Elliott-Evans realization of the irrational rotation algebras as direct limits of circle algebras [EE]; we use this result to derive Theorem B from Theorem A.

The classification program is the analog for C^*-algebras of the classification of hyperfinite factors in the theory of von Neumann algebras. It has only recently become apparent that some reasonable class of simple C^*-algebras is in fact classifiable. Classification of separable commutative C^*-algebras is equivalent to classification of second countable locally compact Hausdorff spaces, a problem long considered out of reach, while classification of commutative weak* separable von Neumann algebras is fairly easy. Similarly, even ignoring the problem of classifying spaces, type I C^*-algebras are very much harder to classify than type I von Neumann algebras. (The extension problem remains a major obstacle, even in the presence of KK-theory.) These differences between the C^*-algebra and von Neumann algebra situations have led to the assumption that simple C^*-algebras should be much harder to classify than factors. (For example, in [Ph1], the second author asked for a proof that $A_{\theta_1} \otimes \mathcal{O}_m \not\cong A_{\theta_2} \otimes \mathcal{O}_m$ for most values of θ_1 and θ_2.) The results of the classification program described above, including those of this paper, suggest that simple C^*-algebras are in fact classifiable, at least in the "amenable" case (or perhaps a large subset of it).

Independently of this work, Rørdam has in Section 5 of [Rr3] introduced a "classifiable class" of separable unital purely infinite simple C^*-algebras. (The main object of Rørdam's paper is the classification of certain purely infinite simple C^*-algebras A for which $K_1(A)$ is torsion-free; the K_1-groups of the algebras in this paper are all odd torsion groups.) Algebras A in Rørdam's class are determined up to isomorphism by the invariant $(K_0(A), [1_A], K_1(A))$. We certainly believe that our algebras are in fact in his class. Unfortunately, neither Rørdam's methods nor ours seem to prove this. In particular, his results do not seem to help with the proof of our Theorem B, the isomorphism of tensor products of different irrational rotation algebras with the same even Cuntz algebra, or other similar results in our Section 5.

Bratteli, Elliott, Evans, and Kishimoto are presently working on a classification theorem presumably covering a larger class of simple C^*-algebras.

Our proof follows the standard outline, first introduced by Elliott in [Ell2]. (We use KK-classes in place of homomorphisms on K-theory, as in [Rr2].) Thus, we construct approximate intertwinings of two given direct systems, and for this we need an existence theorem and a uniqueness theorem. The first three sections of this paper contain the uniqueness theorem, the fourth contains the existence theorem, and in the last section we put everything together. Our uniqueness theorem would obviously have been impossible without Rørdam's results on even Cuntz algebras [Rr1]. In fact, we need his results not only in Section 3, but also for the preliminary work in Sections 1 and 2, and even for the existence theorem in Section 4. However, no proofs in this paper closely resemble the proof of the main technical result in [Rr1].

In more detail, the outline of this paper is as follows. In Section 1 we define approximately absorbing homomorphisms, and give some conditions under which homomorphisms from $C(S^1) \otimes \mathcal{O}_m$ to a purely infinite simple C^*-algebra are automatically approximately absorbing. It turns out that we need the main technical theorem of [Rr1] to prove that anything at all is approximately absorbing. In Section 2 we prove a weak version, sufficient for our purposes, of the following statement: If m is even, then two homomorphisms from $C(S^1) \otimes \mathcal{O}_m$ to a purely infinite simple C^*-algebra A, with the same class in $KK^0(C(S^1) \otimes \mathcal{O}_m, A)$, can be connected by a "discrete homotopy" of asymptotic morphisms. This section is the longest in the paper; it also requires [Rr1]. Section 3 contains the proof of the uniqueness theorem. It is based on the absorption argument first introduced by the second author in [Ph2], and requires results of the first author [Ln3] on approximately commuting unitaries to cope with the fact that Section 2 yields only asymptotic morphisms. It is this absorption argument that requires that homomorphisms be assumed approximately absorbing. In Section 4 we prove the existence theorem: every class in $KK^0(C(S^1) \otimes \mathcal{O}_m, C(S^1) \otimes \mathcal{O}_n)$, with m, n even,

is represented by a homomorphism, unital if the K-theory data allows it. We use the Universal Coefficient Theorem to combine various homomorphisms already constructed in the literature. Here, too, we need to restrict to even Cuntz algebras, since the construction, by Loring [Lr2], of one of the needed homomorphisms requires Rørdam's classification theorem in [Rr1]. Finally, in Section 5 we put the existence and uniqueness theorems together to prove our main theorem and derive various corollaries. We also prove several auxiliary results, such as Theorem C, and construct the co-Cuntz algebras $\mathcal{Q}_m$.

We state here some terminology and notation that will be used throughout this paper.

0.1 Definition A *Cuntz-circle algebra* is a C^*-algebra A which has the form

$$A \cong \bigoplus_{i=1}^{k} M_{r(i)} \otimes C(X_i) \otimes \mathcal{O}_{m(i)},$$

with each X_i being a connected compact subset of the circle S^1. In this expression, k and the $m(i)$ are finite, and $m(i) \geq 2$. An *even* Cuntz-circle algebra is one for which all the $m(i)$ above are even.

Cuntz-circle algebras are thus finite direct sums of matrix algebras over $C(X) \otimes \mathcal{O}_m$, where X is allowed to be homeomorphic to a circle, a point, or a compact interval.

0.2 Conventions (1) Throughout this paper, u is the canonical generating unitary in $C(S^1)$, or in $C(X)$ for any subset $X \subset S^1$. Also, $s_1, \ldots, s_m$ are the canonical generating isometries of the Cuntz algebra $\mathcal{O}_m$; they thus satisfy

$$s_j^* s_j = 1 \quad \text{and} \quad \sum_{j=1}^{m} s_j s_j^* = 1.$$

(2) Let B be a C^*-algebra. We write B^+ for the unitization of B (the unit being added regardless of whether or not B already has one). We write $\tilde{B}$ for the algebra which is equal to B if B is unital, and equal to B^+ if B is not unital.

(3) If B is a unital C^*-algebra, then $U(B)$ is the unitary group of B and $U_0(B)$ is the connected component of $U(B)$ containing the identity of B.

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1 Approximately absorbing homomorphisms

In this section, we introduce approximately absorbing homomorphisms and give some of their elementary properties. We have not proved that homomorphisms from $C(S^1) \otimes \mathcal{O}_m$ to a purely infinite simple C^*-algebra B are automatically approximately absorbing, so this condition appears as a hypothesis in our most general theorems. However, we will see in this section that the homomorphisms in the direct systems corresponding to the most interesting cases (tensor products of even Cuntz algebras with irrational rotation algebras, Bunce-Deddens algebras, etc.) are automatically approximately absorbing.

Rørdam's work ([Rr1] and [Rr2]) is already needed to prove that a homomorphism from $\mathcal{O}_m$ to B is approximately absorbing. We will therefore need to assume throughout this section that our Cuntz algebras are even.

We begin by establishing terminology and notation for approximate unitary equivalence.

1.1 Definition. Let A and B be C^*-algebras, let G be a set of generators of A, and let φ and ψ be two homomorphisms from A to B. We say that φ and ψ are *approximately unitarily equivalent to within* ε, with respect to G, if there is a unitary $v \in \tilde{B}$ such that

$$\|\varphi(g) - v\psi(g)v^*\| < \varepsilon$$

for all $g \in G$. We abbreviate this as

$$\varphi \overset{\varepsilon}{\sim} \psi.$$

(Note that we have suppressed G in the notation.) We say that φ and ψ are *approximately unitarily equivalent* if $\varphi \overset{\varepsilon}{\sim} \psi$ for all $\varepsilon > 0$. (Of course, this notion does not depend on the choice of G.)

1.2 Convention In the previous definition, if $A = C(S^1) \otimes \mathcal{O}_m$ (or $C(X) \otimes \mathcal{O}_m$ with $X \subset S^1$), then we will take the generating set G to be

$$\{u \otimes 1\} \cup \{1 \otimes s_j : j = 1, \ldots, m\},$$

unless otherwise specified.

1.3 Definition Let A be any unital C^*-algebra, and let B be a purely infinite simple C^*-algebra. Let $\varphi, \psi : A \to B$ be two homomorphisms, and assume that $\varphi(1) \neq 0$ and $[\psi(1)] = 0$ in $K_0(B)$. We define a homomorphism $\varphi \widetilde{\oplus} \psi : A \to B$, well defined up to unitary equivalence, by the following construction. Choose a projection $q \in B$ such that $0 < q < \varphi(1)$ and $[q] = 0$. Since B is purely infinite and simple, there are partial isometries v and w such that $vv^* = \varphi(1) - q$, $v^*v = \varphi(1)$, $ww^* = q$, and $w^*w = \psi(1)$. Now define $(\varphi \widetilde{\oplus} \psi)(a) = v\varphi(a)v^* + w\psi(a)w^*$ for $a \in A$.

1.4 Definition Let X be a compact space, let B be a purely infinite simple C^*-algebra, and let $\varphi : C(X) \otimes M_k(\mathcal{O}_m) \to B$ be a homomorphism. Let $X_0 \subset X$ be the closed set such that $\ker(\varphi) = C_0(X \setminus X_0) \otimes M_k(\mathcal{O}_m)$. Then φ is *approximately absorbing* if φ is approximately unitarily equivalent to $\varphi \widetilde{\oplus} \psi$ for any homomorphism $\psi : C(X) \otimes M_k(\mathcal{O}_m) \to B$ of the form $\psi(f \otimes a) = \sum_{i=1}^{l} f(x_i)\psi_i(a)$, where:

(1) $x_1, \ldots, x_l \in X_0$.

(2) $\psi_1, \ldots, \psi_l : M_k(\mathcal{O}_m) \to B$ are homomorphisms.

(3) $\psi_1(1), \ldots, \psi_l(1)$ are mutually orthogonal projections in B.

(4) $[\psi_1] = \cdots = [\psi_l] = 0$ in $KK^0(\mathcal{O}_m, B)$.

Note that, if we regard $C(X) \otimes M_k(\mathcal{O}_m)$ as the C^*-algebra of $M_k(\mathcal{O}_m)$-valued continuous functions on X, then, for $f \in C(X) \otimes M_k(\mathcal{O}_m)$, we may also write $\psi(f) = \sum_{i=1}^{l} \psi_i(f(x_i))$.

If A is a Cuntz-circle algebra, then we say that $\varphi : A \to B$ is approximately absorbing if the restriction of φ to each summand of A is approximately absorbing.

The next lemmas will be about approximately absorbing homomorphisms from $C(X) \otimes M_k(\mathcal{O}_m)$, but they have trivial generalizations to homomorphisms from Cuntz-circle algebras.

1.5 Remark It is clear from the definition that $\varphi : C(X) \otimes M_k(\mathcal{O}_m) \to B$ is approximately absorbing if and only if φ is approximately absorbing when regarded as a homomorphism from $C(X) \otimes M_k(\mathcal{O}_m)$ to $\varphi(1)B\varphi(1)$.

The following lemma asserts that homomorphisms are approximately absorbing when X is a point and m is even.

1.6 Lemma Let m be even, let B be a purely infinite simple C^*-algebra, and let $\varphi,\ \psi : M_k(\mathcal{O}_m) \to B$ be two homomorphisms. If $\varphi \neq 0$ and $[\psi] = 0$ in $KK(M_k(\mathcal{O}_m), B)$, then φ is approximately unitarily equivalent to $\varphi \widetilde{\oplus} \psi$.

Proof: Since B is purely infinite, $\psi(1)$ is equivalent to a subprojection of $\varphi(1)$. We may therefore assume $\psi(1) \leq \varphi(1)$. Replacing B by $\varphi(1)B\varphi(1)$, we may assume that B and φ are unital. Since B is purely infinite and $M_k(\mathcal{O}_m)$ is a unital direct limit of even Cuntz algebras, the result now follows from Theorem 5.3 of [Rr2]. ∎

The next lemma provides a more convenient way to show that a homomorphism is approximately absorbing.

1.7 Lemma Let the notation be as in Definition 1.4, with $X \subset S^1$ and m even. Then φ is approximately absorbing if and only if for every $\varepsilon > 0$ and $\lambda_1, \ldots, \lambda_l \in X_0$, there exist a unital homomorphism $\sigma : M_k(\mathcal{O}_m) \to \varphi(1)B\varphi(1)$, a unitary $v \in \varphi(1)B\varphi(1)$, and nonzero mutually orthogonal projections $q_1, \ldots, q_l \in \varphi(1)B\varphi(1)$ such that:

(1) For $j = 1, \ldots, m$, and for each standard matrix unit $e \in M_k$,

$$\|\sigma(e \otimes s_j) - \varphi(1 \otimes e \otimes s_j)\| < \varepsilon.$$

(2) $\|v - \varphi(u \otimes 1)\| < \varepsilon$.

(3) v commutes with the range of σ and with the q_i, and each q_i commutes with the range of σ.

(4) $q_i v q_i = \lambda_i q_i$ for $i = 1, \ldots, l$.

Proof: Note that the conditions on σ and v say that the map $f \otimes a \mapsto f(v)\sigma(a)$ defines a homomorphism from $C(X) \otimes M_k(\mathcal{O}_m)$ to B which agrees with φ to within ε on a particular set G of generators.

Replacing B by $\varphi(1)B\varphi(1)$, we may assume that B and φ are unital.

Suppose φ is approximately absorbing. Choose nonzero mutually orthogonal projections $p_1, \ldots, p_l \in B$ whose classes in $K_0(B)$ are all zero. With the help of km mutually orthogonal projections summing to p_i whose K_0-classes are zero, it is easy to construct a homomorphism $\psi_i : M_k(\mathcal{O}_m) \to B$ such that $\psi_i(1) = p_i$. Define $\psi : C(X) \otimes M_k(\mathcal{O}_m) \to B$ by $\psi(f \otimes a) = \sum_i f(\lambda_i)\psi_i(a)$. Since φ is approximately absorbing, φ is approximately unitarily equivalent to $\varphi \widetilde{\oplus} \psi$. Let w implement this approximate unitary equivalence to within ε on the generators listed in the statement of the lemma, and set $v = w(\varphi \widetilde{\oplus} \psi)(u)w^*$ and $q_i = w p_i w^*$.

Conversely, let the conditions of the lemma hold for φ, and let ψ be as in the definition of approximately absorbing. We follow the notation of that definition, except that we call the points λ_i instead of x_i. Let $\varepsilon > 0$. We want to show that $\varphi \overset{\varepsilon}{\sim} \varphi \widetilde{\oplus} \psi$ (with respect to the set G of generators above). Choose a

homomorphism σ, a unitary v, and projections q_i, as in the hypotheses, except using $\varepsilon/3$ in place of ε. Define $\varphi' : C(X) \otimes M_k(\mathcal{O}_m) \to B$ by $\varphi'(f \otimes a) = f(v)\sigma(a)$. Then $\varphi' \overset{\varepsilon/3}{\sim} \varphi$, whence also $\varphi' \widetilde{\oplus} \psi \overset{\varepsilon/3}{\sim} \varphi \widetilde{\oplus} \psi$. It therefore suffices to show that $\varphi' \widetilde{\oplus} \psi \overset{\varepsilon/3}{\sim} \varphi'$.

To see this, let $q_0 = \varphi'(1) - q_1 - \cdots - q_l$, define $\varphi'_i(f \otimes a) = f(\lambda_i) q_i \sigma(a)$, and observe that $\varphi'(b) = \sum_{i=0}^{l} \varphi'_i(b)$. Further define $\overline{\psi}_i(f \otimes a) = f(\lambda_i)\psi_i(a)$, so that $\psi(b) = \sum_{i=1}^{l} \overline{\psi}_i(b)$. It follows from the previous lemma that φ'_i is approximately unitarily equivalent to $\varphi'_i \widetilde{\oplus} \overline{\psi}_i$. Forming the (orthogonal) sum over $i = 1, \ldots, l$, and adding φ'_0, gives the required approximate unitary equivalence. ∎

1.8 Corollary Let m be even. Let B and C be purely infinite simple C^*-algebras, and let $\eta : B \to C$ be a nonzero homomorphism. If $X \subset S^1$ and $\varphi : C(X) \otimes M_k(\mathcal{O}_m) \to B$ is approximately absorbing, then $\eta \circ \varphi$ is also approximately absorbing.

Proof: Since B is simple, η is injective, and the condition of the lemma is preserved under application of η. ∎

1.9 Corollary Let A be a simple separable C^*-algebra, obtained as a simple direct limit with no dimension growth in the sense of [Ph3]. Let m be even, let B be a purely infinite simple C^*-algebra, and let $X \subset S^1$ be compact. Let $\iota : A \otimes \mathcal{O}_m \to B$ be an injective homomorphism, and let $\mu : C(X) \otimes M_k \to A$ be a nonzero homomorphism. Then $\iota \circ (\mu \otimes \mathrm{id}_{\mathcal{O}_m})$ is approximately absorbing.

Proof: By the lemma, it suffices to show that for $\varepsilon > 0$ and $\lambda_1, \ldots, \lambda_l \in X$, there exists unitary v and nonzero mutually orthogonal projections $q_1, \ldots, q_l$ in $\mu(1)A\mu(1)$, all commuting with $\mu(1 \otimes M_k)$, such that $\|v - \mu(u \otimes 1)\| < \varepsilon$ and $q_i v q_i = \lambda_i q_i$ for each i. (The required objects in B are then gotten by tensoring with the identity in $\mathcal{O}_m$ and applying ι.) It actually suffices to choose a rank one projection $e \in M_k$, and approximate $\mu(u \otimes e)$ by a unitary with this property in $\mu(e)A\mu(e)$. We can then ignore the commutant condition.

If $X = S^1$ and $\mathrm{sp}(\mu(u \otimes e)) = S^1$ (evaluated in $\mu(e)A\mu(e)$), and if $\mu(e)A\mu(e)$ is again a simple direct limit with no dimension growth, then the desired approximation is now Lemma 5.2 of [Ph3]. The proof of the general case uses essentially the same reasoning as the proof of that lemma. ∎

1.10 Corollary Let m be even and let B be a purely infinite simple C^*-algebra. Let $X \subset S^1$ be compact, let $\varphi : C(X) \otimes \mathcal{O}_m \to B$ be a homomorphism, and let $k \geq 1$. Then φ is approximately absorbing if and only if $\mathrm{id}_{M_k} \otimes \varphi : C(X) \otimes M_k(\mathcal{O}_m) \to M_k(B)$ is approximately absorbing.

Proof: Let φ be approximately absorbing, and let $X_0 \subset X$ be as before. Let $\varepsilon > 0$ and $\lambda_1, \ldots, \lambda_l \in X_0$. Choose a homomorphism $\sigma_0 : \mathcal{O}_m \to B$, a unitary v_0, and projections $q_i^{(0)}$ for φ as in the condition of the lemma. Then set $\sigma = \mathrm{id}_{M_k} \otimes \sigma_0$, $q_i = 1 \otimes q_i^{(0)}$, and $v = 1 \otimes v_0$. This shows that $\mathrm{id}_{M_k} \otimes \varphi$ is approximately absorbing.

Conversely, let $\mathrm{id}_{M_k} \otimes \varphi$ be approximately absorbing, and let $\varepsilon > 0$ and $\lambda_1, \ldots, \lambda_l \in X_0$. Without loss of generality we may assume the λ_i are distinct. Let $\{e_{\mu\nu}\}$ be a system of matrix units in M_k. Choose $\delta > 0$ such that if $\{f_\mu\}$ are projections in $M_k(B)$ such that $\|e_{\mu\mu} \otimes 1 - f_\mu\| < \delta$, then there is a unitary $z \in M_k(B)$ such that $z f_\mu z^* = e_{\mu\mu} \otimes 1$ and $\|z - 1\| < \varepsilon/3$. Now choose $\sigma : M_k(\mathcal{O}_m) \to M_k(B)$ and $v, q_1, \ldots, q_l \in M_k(B)$ for $\mathrm{id}_{M_k} \otimes \varphi$ as in Lemma 1.7, using $\min(\delta, \varepsilon/3)$ for ε, and taking the standard matrix units to be $\{e_{\mu\nu}\}$. Replace σ, v, and $q_1, \ldots, q_l$ by their conjugates by z. This gives σ, v, and $q_1, \ldots, q_l$ as in the lemma, with norm estimate ε, and with $\sigma(e_{\mu\mu} \otimes 1) = e_{\mu\mu} \otimes 1$.

Now set $\sigma_0(a) = \sigma(e_{11} \otimes a)$ for $a \in \mathcal{O}_m$, and set $v_0 = e_{11} v e_{11}$. For fixed i, choose μ such that $(e_{\mu\mu} \otimes$

$1)q_i(e_{\mu\mu} \otimes 1) \neq 0$, and set

$$q_i^{(0)} = (\mathrm{id}_{M_k} \otimes \varphi)(e_{1\mu} \otimes 1)q_i(\mathrm{id}_{M_k} \otimes \varphi)(e_{\mu 1} \otimes 1).$$

It is easy to check the commutation relations and norm estimates required to satisfy the conditions of Lemma 1.7. (That the $q_i^{(0)}$ are mutually orthogonal follows from the relations $q_i^{(0)} v_0 = v_0 q_i^{(0)} = \lambda_i q_i^{(0)}$ and the fact that the λ_i are distinct.) This shows that φ is approximately absorbing. ∎

1.11 Lemma Let $X \subset S^1$ be compact and connected, and let $p \in A = C(X) \otimes M_n(\mathcal{O}_m)$ be a nonzero projection. Then there are $l \leq k$ and an isomorphism $C(X) \otimes M_k(\mathcal{O}_m) \cong C(X) \otimes M_n(\mathcal{O}_m)$ which sends $C(X) \otimes M_l(\mathcal{O}_m)$, regarded as a corner in $C(X) \otimes M_k(\mathcal{O}_m)$, onto pAp.

Proof: We identify A with $C(X, M_n(\mathcal{O}_m))$. Theorem B of [Zh3] implies that p is homotopic to, and hence unitarily equivalent to, a constant projection q. (If $X = S^1$, this uses the fact that $K_1(\mathcal{O}_m) = 0$.) Conjugating by this unitary, we may assume that p itself is a constant projection. This reduces us to consideration of the case X is a point. Thus, we assume $A = M_n(\mathcal{O}_m)$. Without loss of generality, we also assume $p \neq 1$.

Since $K_0(\mathcal{O}_m)$ is finite cyclic and generated by $[1]$, we can find $0 < l < k$ such that $l[1] = [p]$ and $k[1] = n[1]$ in $K_0(\mathcal{O}_m)$. Since $A = M_n(\mathcal{O}_m)$ is purely infinite and simple, there are l orthogonal projections in A, each Murray-von Neumann equivalent to $1_{\mathcal{O}_m}$, which sum to p, and $k - l$ such projections which sum to $1 - p$. Since all k projections are Murray-von Neumann equivalent to $1_{\mathcal{O}_m}$, they induce an isomorphism $M_k(\mathcal{O}_m) \cong A$ which sends $M_l(\mathcal{O}_m)$ onto pAp. ∎

1.12 Lemma Let B be a purely infinite simple C^*-algebra, let $X \subset S^1$ be compact and have finitely many connected components, and let m be even. If $\varphi : C(X) \otimes M_k(\mathcal{O}_m) \to B$ is approximately absorbing, and $p \in C(X) \otimes M_k(\mathcal{O}_m)$ is a nonzero projection, then $\varphi_0 = \varphi|_{p[C(X) \otimes M_k(\mathcal{O}_m)]p}$ is also approximately absorbing.

Proof: We first prove this in the special case in which p is the characteristic function χ_Y of some closed and open subset $Y \subset X$.

Let F be a generating set for $C(Y) \otimes M_k(\mathcal{O}_m)$, which we regard as a subalgebra of $C(X) \otimes M_k(\mathcal{O}_m)$, and let G be a generating set for $C(X) \otimes M_k(\mathcal{O}_m)$ which contains F and $u \otimes 1$. Let $\varepsilon > 0$, and let $\psi : C(Y) \otimes M_k(\mathcal{O}_m) \to B$ be as in Definition 1.4. When necessary, regard ψ as defined on all of A by setting it equal to zero on $(1 - p)A(1 - p)$. Choose $\delta > 0$ such that if w_1 and w_2 are unitaries in a C^*-algebra with spectrum contained in X, and $\|w_1 - w_2\| < \delta$, then the projections $\chi_Y(w_1)$ and $\chi_Y(w_2)$ are so close that there is a unitary z which conjugates one to the other and satisfies $\|z - 1\| < \varepsilon/4$. Let $\rho = \min(\delta, \varepsilon/2)$. Since φ is approximately absorbing, we have $\varphi \widetilde{\oplus} \psi \overset{\rho}{\sim} \varphi$ with respect to G. Conjugating the first of these by a suitable unitary, we may assume they actually agree to within ρ on G. Now, with $w_1 = (\varphi \widetilde{\oplus} \psi)(u \otimes 1)$ and $w_2 = \varphi(u \otimes 1)$, conjugate $\varphi \widetilde{\oplus} \psi$ by z as above. The resulting homomorphisms agree exactly on $p = \chi_Y(u \otimes 1)$ and to within $2\varepsilon/4 + \rho \leq \varepsilon$ on F. Therefore the cutdowns by $\varphi(p)$ agree to within ε on F. These cutdowns are $\varphi_0 \widetilde{\oplus} \psi$ and φ_0 respectively. So the special case is proved.

Combining the special case just proved with Remark 1.5, we see that it now suffices to prove the lemma when X is connected. Lemma 1.11 reduces this proof to two applications of Corollary 1.10. ∎

To deal with maps from Cuntz-circle algebras to nonsimple Cuntz-circle algebras, we introduce the following definition. We include the injectivity condition because it is needed in Section 5.

1.13 Definition Let D be any C^*-algebra, let X be compact, and let $\varphi : C(X) \otimes \mathcal{O}_m \to D$ be a homomorphism. Then φ is *permanently approximately absorbing* if whenever B is a purely infinite simple C^*-algebra and $\mu : D \to B$ is any nonzero homomorphism, then $\mu \circ \varphi$ is injective and approximately absorbing.

To see that the definition is not vacuous, note that any injective approximately absorbing homomorphism

to a purely infinite simple C^*-algebra satisfies the conditions.

The next two lemmas provide all the permanently approximately absorbing homomorphisms we will need in this paper.

1.14 Lemma Let A be a Cuntz-circle algebra, and let $\varphi : A \to D$ be a permanently approximately absorbing homomorphism. Let E be a nonzero C^*-algebra, and let $\psi : D \to E$ be any homomorphism such that $\psi(D)$ is not contained in any proper ideal of E. Then $\psi \circ \varphi$ is again permanently approximately absorbing.

Proof: Let B be a purely infinite simple C^*-algebra, and let $\eta : E \to B$ be a nonzero homomorphism. Then $\eta \circ \psi \neq 0$, so we apply the definition to φ. ∎

1.15 Lemma Let m be even, let B be a purely infinite simple C^*-algebra, and let Y be a compact Hausdorff space. Let X be either S^1, a closed arc in S^1, or a point in S^1. Then there exists a permanently approximately absorbing homomorphism $\varphi : C(X) \otimes M_n(\mathcal{O}_m) \to C(Y) \otimes B$ such that $[\varphi] = 0$ in $KK^0(C(X) \otimes M_n(\mathcal{O}_m), C(Y) \otimes B)$, and such that for every $\varepsilon > 0$ and any finite subset $G \subset C(X) \otimes M_n(\mathcal{O}_m)$, there is a homomorphism $\psi : C(X) \otimes M_n(\mathcal{O}_m) \to C(Y) \otimes B$ satisfying:

(1) $\|\psi(g) - \varphi(g)\| < \varepsilon$ for all $g \in G$.

(2) There are $x_1, \ldots, x_l \in X$ and homomorphisms $\psi_1, \ldots, \psi_l$ from $M_n(\mathcal{O}_m)$ to $C(Y) \otimes B$, whose classes in $KK^0(M_n(\mathcal{O}_m), C(Y) \otimes B)$ are all zero, such that $\psi_1(1), ..., \psi_l(1)$ are mutually orthogonal projections and such that $\psi(f \otimes a) = \sum_{i=1}^{l} f(x_i)\psi_i(a)$ for $f \in C(X)$ and $a \in M_n(\mathcal{O}_m)$.

(3) $\psi(1) = \varphi(1)$.

If $[1] = 0$ in $K_0(B)$, then φ may be chosen unital.

Proof: By Corollary 1.10, we need only consider the case $n = 1$. (To see that this applies to the last statement, assume $[1] = 0$ in $K_0(B)$. Then 1_B is Murray-von Neumann equivalent to $1_{M_n(B)}$ in $M_n(B)$. Therefore $B \cong M_n(B)$, so that a unital homomorphism to $C(Y) \otimes M_n(B)$ is the same as one to $C(Y) \otimes B$.)

Next, we observe that it suffices to prove the lemma with $B = \mathcal{O}_2$, with Y a one point space, and merely requiring injectivity in place of the permanently approximately absorbing condition. To see this, suppose that $\varphi_0 : C(X) \otimes \mathcal{O}_m \to \mathcal{O}_2$ has the required properties. It is easy to find a nonzero (hence injective) homomorphism $\sigma_0 : \mathcal{O}_2 \to B$, unital if $[1] = 0$ in $K_0(B)$. Define $\sigma : \mathcal{O}_2 \to C(Y) \otimes B$ by $\sigma(a) = 1 \otimes \sigma_0(a)$, and set $\varphi = \sigma \circ \varphi_0$. Then $[\varphi] = [\sigma] \times [\varphi_0] = 0$ in $KK^0(C(X) \otimes \mathcal{O}_m, C(Y) \otimes B)$. It is furthermore clear that for every $\varepsilon > 0$ and finite $G \subset C(X) \otimes \mathcal{O}_m$ there is a homomorphism ψ as required in the lemma. Finally, if C is some other purely infinite simple C^*-algebra, and $\lambda : C(X) \otimes \mathcal{O}_m \to C(Y) \otimes B$ is any nonzero homomorphism, then $\lambda \circ \sigma$ is nonzero. Since $\mathcal{O}_2$ is simple, $\lambda \circ \sigma$ is injective. The condition on the existence of the map ψ for φ_0, combined with an easy argument using Lemma 1.7, now shows that $\lambda \circ \varphi$ is approximately absorbing. Thus φ is permanently approximately absorbing. This completes the reduction to the case $B = \mathcal{O}_2$ etc.

Since $K_*(\mathcal{O}_2) = 0$, the Universal Coefficient Theorem [RS] implies that $KK^0(\mathcal{O}_2, \mathcal{O}_2) = 0$. Therefore $KK^0(A, \mathcal{O}_2) = 0$ for any separable nuclear C^*-algebra A. Thus, in the special case $B = \mathcal{O}_2$, we can ignore the requirement that homomorphisms be zero in KK-theory.

Let D be the 2^∞ UHF algebra. We show that if $X \subset S^1$ is as in the lemma, then D contains a unitary v whose spectrum is X. If X is a point, this is trivial. Otherwise, note that D is a non-elementary simple C^*-algebra. By page 61 of [AS], there exists a selfadjoint $h \in D$ such that $\mathrm{sp}(h) = [0, 1]$. Exponentiating, we obtain a unitary whose spectrum is S^1 or any given closed arc in S^1.

Define $\mu : C(X) \to D$ by $\mu(u) = v$. Further choose a unital homomorphism $\tau : \mathcal{O}_m \to \mathcal{O}_2$. (It is easy to find

such a thing.) Note that $D \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ by [Rr1]. Define φ to be the composite

$$C(X) \otimes \mathcal{O}_m \xrightarrow{\mu \otimes \tau} D \otimes \mathcal{O}_2 \xrightarrow{\cong} \mathcal{O}_2.$$

The approximating maps ψ are obtained by replacing $\mu(u)$ by approximating unitaries with finite spectrum, which exist since D is a UHF algebra. ∎

2 Homotopies of asymptotic morphisms

Results of Dădărlat and Loring [DL] imply that if $\varphi_0, \varphi_1 : C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m \to B$ are two homomorphisms with the same class in $KK^0(C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m, B)$, and if B is stable, then φ_0 and φ_1 are homotopic via asymptotic morphisms. (Asymptotic morphisms were defined in [CH].) We are going to need a similar statement for homomorphisms from the unital C^*-algebra $C(S^1) \otimes \mathcal{O}_m$. Of course, some conditions will have to be imposed, some obvious (both homomorphisms nonzero), some not so obvious. Furthermore, we don't actually need a homotopy of asymptotic morphisms. What we need, and what we construct, is weaker in two ways. First, for each value α of the homotopy parameter, instead of an asymptotic morphism we merely produce a single linear map which is multiplicative to within a prespecified $\varepsilon > 0$ on our standard set of generators. Second, the homotopy we construct does not agree exactly with the original maps at the endpoints, but only to within ε on the generators. This section, the longest one in the paper, is devoted to producing such a thing under suitable conditions. This is done in Lemma 2.10.

We will need to know that two nonunital nonzero homomorphisms from $\mathcal{O}_m$ to B, having the same class in KK-theory, are homotopic. This requires Rørdam's results [Rr1], so we will have to take m even. We will also have to assume B is a purely infinite simple C^*-algebra (although not necessarily stable). And we will need some other technical conditions. Lemma 2.11, the last lemma in this section, shows how to force some of them to hold.

The first part of this section is devoted to showing that certain hereditary subalgebras of $C([0,1], B)$ have increasing approximate identities consisting of projections. This fact is an essential technical step in the proof of Lemma 2.10. In fact, the hereditary subalgebras we consider can be shown, with only a few more paragraphs, to be isomorphic to $C([0,1], B) \otimes \mathcal{K}$.

It is probably true that, under suitable conditions, two homomorphisms from $C(S^1) \otimes \mathcal{O}_m$ to B with the same class in $KK^0(C(S^1) \otimes \mathcal{O}_m, B)$ are homotopic via asymptotic morphisms. The conditions should be: B purely infinite simple, m even (at least for now), and both homomorphisms nonunital and "absorbing up to homotopy" in a suitable sense. We don't need such a strong result, and so we don't try to prove it, but this should be contrasted with the paper of Dădărlat and Loring [DL], in which the domain algebra is never unital. Note that Rørdam's work easily implies such a result for $\mathcal{O}_m$ in place of $C(S^1) \otimes \mathcal{O}_m$, even using homomorphisms instead of asymptotic morphisms. (See Lemma 2.9.)

We begin by summarizing in a convenient form some standard approximation results in C^*-algebras.

2.1 Lemma There exist functions β_1, β_2, β_3, β_4, and $\beta_5^{(m)}$ from $[0, \infty)$ to $[0, \infty]$ which are nondecreasing and satisfy $\lim_{t \to 0} \beta_i(t) = 0$, and which provide the following estimates for approximation problems in a general C^*-algebra A. In (5), we write simply β_5 when m is understood. In all parts, it is to be understood that when $\beta_i(\eta) = \infty$, the elements claimed may in fact not exist.

(1) If $p_0 \in A$ is selfadjoint and satisfies $\|p_0^2 - p_0\| < \eta$, then there exists a projection $p \in A$ such that $\|p - p_0\| < \beta_1(\eta)$.

(2) If $p, q \in A$ are projections such that $\|pq - q\| < \eta$, then there exists a projection $p' \in A$ such that $p' \geq q$ and $\|p' - p\| < \beta_2(\eta)$.

(3) If $p, q \in A$ are projections such that $\|pq - q\| < \eta$ (as in (2)), then there exists a projection $q' \in A$ and a unitary path $t \mapsto v_t \in \widetilde{A}$ such that $p \geq q'$, $\|q' - q\| < \beta_3(\eta)$, $v_0 = 1$, $v_1 q' v_1^* = q$, and $\|v_t - 1\| < \beta_3(\eta)$ for all t.

(4) If $p, q \in A$ are projections and $s_0 \in A$ satisfies $\|p - s_0^* s_0\| < \eta$ and $\|q - s_0 s_0^*\| < \eta$, then there is a partial isometry $s \in A$, given by $s = (p s_0 q)[(p s_0 q)^*(p s_0 q)]^{-1/2}$ (functional calculus evaluated in qAq), which satisfies $s^* s = p$, $s s^* = q$, and $\|s - s_0\| < \beta_4(\eta)$.

(5) If A is unital, and s_j, $w_j \in A$ (for $j = 1, \ldots, m$) are partial isometries such that $s_j^* s_j = 1$, $\sum_i s_i s_i^* = 1$, the projections $q_j = w_j w_j^*$ and $q = \sum_i q_i$ satisfy $w_j^* w_j = q$ for each j, and

$$\|q_j s_j q - w_j\| < \eta,$$

then there exist partial isometries y_j such that $y_j^* y_j = \sum_i y_i y_i^* = 1 - q$ and

$$\|s_j - (w_j + y_j)\| < \beta_5^{(m)}(\eta).$$

Proof: Parts (1)–(4) are well known and have standard proofs, which we omit. We sketch the proof of (5).

We start by observing that

$$\|q s_j^* (1-q) s_j q\| \leq \|q s_j^* (1-q_j) s_j q\| = \|q - q s_j^* q_j s_j q\| < 2\eta.$$

Therefore

$$\|(1-q) s_j q\| < (2\eta)^{1/2}.$$

We also estimate $\|q s_j (1-q)\|$. We have

$$\|s_k q - q_k s_k q\| = \|(s_k q - q_k s_k q)^* (s_k q - q_k s_k q)\|^{1/2} = \|q - q s_k^* q_k s_k q\|^{1/2} < (2\eta)^{1/2}.$$

So

$$\|s_k q - w_k\| < \eta + (2\eta)^{1/2}.$$

Therefore

$$\|q s_j (1-q)\| = \left\| \sum_{k=1}^m w_k w_k^* s_j (1-q) \right\| \leq 2m(\eta + (2\eta)^{1/2}) + \left\| \sum_{k=1}^m s_k q_k s_k^* s_j (1-q) \right\|.$$

The last term is equal to $\|s_j q_j s_j^* s_j (1-q)\| = 0$.

The estimates on $\|(1-q) s_j q\|$ and $\|q s_j (1-q)\|$ show that q approximately commutes with each s_j.

The desired result now follows from the fact that the defining relations of $\mathcal{O}_m$ are exactly stable in the sense of Loring [Lr1], [Lr3]. This follows from Theorem 2.6 of [Lr3] and Corollary 2.24 of [Bl1]. (Note that "exactly semiprojective" in [Lr3] is the same as "semiprojective" in [Bl1].) ∎

The following notation will be used throughout this section.

2.2 Notation If B is a C^*-algebra, X is a compact Hausdorff space, and D is a hereditary subalgebra of $C(X, B)$, we let $\mathrm{ev}_x : C(X, B) \to B$ be the evaluation map at $x \in X$, and define

$$D_x = \mathrm{ev}_x(D) = \{b(x) : b \in D\}.$$

2.3 Lemma Let the notation be as in 2.2. Then D_x is a hereditary C^*-subalgebra of B. (In particular, it is closed.) Moreover, if $b \in C(X, B)$ satisfies $b(x) \in D_x$ for all $x \in X$, then $b \in D$.

Proof: D_x is the image of the C^*-algebra D under the homomorphism $\mathrm{ev}_x : C(X, B) \to B$, and is therefore a C^*-subalgebra of B. It remains to prove that D_x is hereditary. So let $b_0 \in D_x$ and $a_0 \in B$ satisfy $0 \leq a_0 \leq b_0$. Choose $b \in D$ such that $b(x) = b_0$; using standard manipulations we may assume that $b \geq 0$. Now $\{b_0^{1/n}\}$ is an approximate identity for the hereditary subalgebra of B generated by b_0; in fact, for any c in this subalgebra, we actually have $b_0^{1/n} c b_0^{1/n} \to c$. Now let $a \in C(X, B)$ be the constant function with value a_0. Then $b^{1/n} a b^{1/n} \in D$, and $(b^{1/n} a b^{1/n})(x) \to a_0$. Therefore $a_0 \in D_x$.

It remains to prove the last statement. Let $\varepsilon > 0$. For each $x \in X$ choose, using an approximate identity for D and the first part of the lemma, an element $d_x \in D$ such that $\|(d_x b)(x) - b(x))\| < \varepsilon/2$. Choose an open set $U_x \subset X$ such that $\|(d_x b)(y) - b(y))\| < \varepsilon$ for $y \in U_x$. Cover X by finitely many of these sets, and let $\{f_i : 1 \leq i \leq n\}$ be a partition of unity subordinate to this open cover. Let $\mathrm{supp}(f_i) \subset U_{x_i}$. Define $d(x) = \sum_i d_{x_i}(x)(f_i b)(x)$. Then $d \in DB$ and $\|d - b\| < \varepsilon$. Therefore $b \in \overline{DB}$. Similarly $b \in \overline{BD}$. Therefore $b \in \overline{DB} \cap \overline{BD} = D$. ∎

2.4 Lemma Let the notation be as in 2.2. Let $Z \subset X$ be closed, and let $x_0 \notin Z$. Let $p \in D$ be a projection, and let $e \in D_{x_0}$ be a projection homotopic to $p(x_0)$ in D_{x_0}. Then there exists a projection $q \in D$ such that $q(x) = p(x)$ for all $x \in Z$, $q(x_0) = e$, and q is homotopic to p.

Proof: Without loss of generality, assume B is unital. Let $t \mapsto v(t)$ be a unitary path in $\widetilde{D_{x_0}} \subset B$ such that $v(0) = 1$, $v(1)p(x_0)v(1)^* = e$, and $v(t) - 1 \in D_{x_0}$ for $t \in [0,1]$. Then v can be regarded as an element of $U_0(C([0,1], \widetilde{D_{x_0}}))$. Since $C([0,1], \widetilde{D}) \to C([0,1], \widetilde{D_{x_0}})$ is surjective, there exists $w \in U_0(C([0,1], \widetilde{D}))$ whose image in $C([0,1], \widetilde{D_{x_0}})$ is v. An easy adjustment allows us to assume that $w(0) = 1$ and $w(t) - 1 \in D$ for all t. Now choose a continuous function $f : X \to [0,1]$ such that $f|_Z = 0$ and $f(x_0) = 1$. Regarding w as a function from $X \times [0,1]$ to B, define $c \in U(C(X,B))$ by $c(x) = w(x, f(x))$, and define $q(x) = c(x)p(x)c(x)^*$. The required homotopy from p to q will be given by $t \mapsto w(x, tf(x))p(x)w(x, tf(x))^*$.

It remains only to show that these projections are actually in D. This follows immediately from the fact that the unitaries used to define them differ by 1 from elements of D, which is a consequence of the last part of the previous lemma. ∎

2.5 Lemma Let B and D be as in Notation 2.2, and assume in addition that $X = [\alpha, \beta]$ is a closed interval, B is separable, purely infinite, and simple, and D_t is nonzero and nonunital for all t. Let $p \in D$, $e_\alpha \in D_\alpha$, and $e_\beta \in D_\beta$ be projections such that, for $i = \alpha, \beta$ we have

$$e_i > p(i) \quad \text{and} \quad [e_i - p(i)] = 0 \text{ in } K_0(B).$$

Then there exists a projection $q \in D$ such that $q \geq p$, $q(\alpha) = e_\alpha$, and $q(\beta) = e_\beta$.

Proof: Without loss of generality, assume $\alpha = 0$ and $\beta = 1$. Further, replace D by $(1-p)D(1-p)$. This allows us to assume that $p = 0$. The new D_t is still nonunital and nonzero. (Note that $p(t)$ can't be an identity for D_t, since D_t doesn't have an identity.)

Each D_t contains a positive element with norm greater than 1. Therefore, for each t the subalgebra D contains a positive element a such that $\|a(t)\| > 1$. Using a partition of unity argument and Lemma 2.3, we can produce $a \in D$ such that $a \geq 0$ and $\|a(t)\| > 1$ for all t. Cutting down using functional calculus, we can assume $\|a(t)\| = 1$ for all t. Choose a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that $\|a(t) - a(t_i)\| < 1/16$ for $t \in [t_{i-1}, t_i]$. Now D_{t_i}, being purely infinite and simple, has real rank 0. Therefore $a(t_i)$ can be approximated by selfadjoint elements with finite spectrum. Since also $1 \in \mathrm{sp}(a(t_i))$, there is a nonzero projection $f_i \in D_{t_i}$ such that $\|f_i a(t_i) - f_i\| < 1/16$. Since $f_i D_{t_i} f_i$ is purely infinite and $K_0(f_i D_{t_i} f_i) = K_0(D_{t_i})$, there is a nonzero projection $f_i' \in f_i D_{t_i} f_i$ such that $[f_i'] = 0$. (See page 188 of [Cu2].) Replacing f_i by f_i', we may assume furthermore that $[f_i] = 0$ in $K_0(D_{t_i})$. We next observe that $x(t) = a(t) f_i a(t)$ satisfies $x(t) \in D_t$ and $\|x(t) - f_i\| < 1/4$ for $t \in [t_{i-1}, t_i]$.

We can therefore apply functional calculus to produce a continuous function q_i from $[t_{i-1}, t_i]$ to the projections in B such that $q_i(t) \in D_t$ and $\|q_i(t) - f_i\| < 1/2$. It follows that the classes of $q_i(t)$ and f_i in $K_0(B)$ are equal, and that $q_i(t) \neq 0$. Since the inclusion of D_t in B induces an isomorphism on K-theory, it follows that $[q_i(t)] = 0$ in $K_0(D_t)$. In particular, $[q_i(t_i)] = [q_{i+1}(t_i)]$. Note that both $q_i(t_i)$ and $q_{i+1}(t_i)$ are nontrivial. Therefore Theorem 1.1 of [Zh2] implies that these two projections are homotopic. Using the previous

lemma, we now construct q on $[0, t_1]$ so that $q(0) = e_0$ and $q(t_1) = q_1(t_1)$, on $[t_1, t_2]$ so that $q(t_1) = q_1(t_1)$ and $q(t_2) = q_2(t_2)$, etc., finishing by constructing q on $[t_{n-1}, \frac{1}{2}(t_{n-1}+1)]$ so that $q(t_{n-1}) = q_{n-1}(t_{n-1})$ and $q(\frac{1}{2}(t_{n-1}+1)) = q_n(\frac{1}{2}(t_{n-1}+1))$ and on $[\frac{1}{2}(t_{n-1}+1), 1]$ so that $q(\frac{1}{2}(t_{n-1}+1)) = q_n(\frac{1}{2}(t_{n-1}+1))$ and $q(1) = e_1$. Since the definitions agree on the points where the intervals overlap, q is in fact continuous. This is the desired projection. ∎

The hypotheses of the following proposition (as well as those of the previous lemma) imply that D is full in $C([0,1], B)$. Therefore D is stably isomorphic to $C([0,1], B)$. In fact, it turns out that D is isomorphic to $C([0,1], B) \otimes K$. (We won't prove this, but it is an easy step from the conclusion of the next proposition.) If we already knew this, the conclusions would be obvious. But we don't know how to prove such an isomorphism except by using these results.

2.6 Proposition Under the hypotheses of the previous lemma, D has an increasing approximate identity consisting of projections.

Proof: We will prove the following claim: For every positive element $a \in D$, every projection $p \in D$, and every $\varepsilon > 0$, there exists a projection $q \in D$ such that

$$\|qa - a\| < \varepsilon \quad \text{and} \quad \|qp - p\| < \varepsilon.$$

Given this, we first observe that the same result holds with the stronger conclusion $q \geq p$ in place of $\|qp - p\| < \varepsilon$. Indeed, with β_2 as in Lemma 2.1, we choose $\delta > 0$ such that $\|a\|(\delta + \beta_2(\delta)) < \varepsilon$, construct q_0 as above using δ in place of ε, and use Lemma 2.1 (2) to replace q_0 by q such that $q \geq p$ and $\|q - q_0\| < \beta_2(\delta)$. Using the stronger conclusion, we construct our approximate identity by induction, letting a and ε run independently through a countable dense subset of the positive part of D and the set $\{\frac{1}{n} : n \in \mathbf{N}\}$.

We now prove the claim. Without loss of generality, assume $\|a\| \leq 1$.

Since D is separable, it has a strictly positive element b. We first prove the following subclaim: there exists a strictly decreasing sequence $\{\alpha_n\}$ of positive real numbers with $\alpha_n \to 0$ such that

$$(\alpha_{n+1}, \alpha_n) \cap \operatorname{sp}(b(t)) \neq \emptyset$$

for all n and t. We construct α_n inductively, starting with $\alpha_0 = 1$. Given $\alpha_n > 0$, we first observe that there is $\beta_t \in \operatorname{sp}(b(t))$ such that $\alpha_n > \beta_t > 0$. (Otherwise, since $b(t)$ is strictly positive in D_t, this algebra would be unital.) Continuity of the spectrum on selfadjoint elements ensures that there is an open set $U_t \subset [0,1]$ such that for $s \in U_t$ we have

$$\left(\frac{\beta_t}{2}, \frac{\beta_t + \alpha_n}{2} \right) \cap \operatorname{sp}(b(s)) \neq \emptyset.$$

Cover $[0,1]$ with finitely many of these sets, say $U_{t_1}, \ldots, U_{t_l}$, and choose α_{n+1} such that

$$0 < \alpha_{n+1} < \min(\alpha_n/2, \beta_{t_1}, \ldots, \beta_{t_l}).$$

This proves the subclaim.

Let $g_k : [0, \infty) \to [0,1]$ be the continuous function which is 0 on $[0, \alpha_{k+1}]$, 1 on $[\alpha_k, \infty)$, and linear on $[\alpha_{k+1}, \alpha_k]$. Note that $g_{k+1} g_k = g_k$, and that the elements $g_k(b)$ form an approximate identity for D.

Choose $\rho > 0$ such that

$$6\beta_3(14\rho) + 16\rho < \varepsilon \quad \text{and} \quad 3\beta_3(14\rho) + 4\rho < 1/2.$$

(Here the function β_3 is from Lemma 2.1 (3).) Choose k so large that

$$\|g_l(b)a - a\| < \rho \quad \text{and} \quad \|g_l(b)p - p\| < \rho$$

for all $l \geq k$. Next, choose a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

of $[0,1]$ such that, for all $t \in [t_{i-1}, t_i]$, we have

$$\|g_l(b)(t) - g_l(b)(t_{i-1})\| < \rho \quad \text{and} \quad \|g_l(b)(t) - g_l(b)(t_i)\| < \rho$$

for $l = k$ and $l = k+3$, and

$$\|a(t) - a(t_i)\| < \rho \quad \text{and} \quad \|p(t) - p(t_i)\| < \rho.$$

The relations $g_{k+2}(b(t_i))g_{k+1}(b(t_i)) = g_{k+1}(b(t_i))$ and $g_{k+1}(b(t_i))g_k(b(t_i)) = g_k(b(t_i))$ imply, using [Bn2] and the fact that D_{t_i} has real rank zero, the existence of a projection $e_i^{(0)} \in D_{t_i}$ such that $g_{k+2}(b(t_i)) \geq e_i^{(0)} \geq g_k(b(t_i))$. The choice of the sequence $\{\alpha_n\}$ implies that $\mathrm{sp}(g_{k+2}(b(t_i))) \cap (0,1) \neq \emptyset$. Since $g_{k+2}(b(t_i))e_i^{(0)} = e_i^{(0)}$, it follows that $e_i^{(0)}$ is not an identity for the hereditary subalgebra E_i of B generated by $g_{k+2}(b(t_i))$. Since this hereditary subalgebra is purely infinite simple, all classes in its K_0-group are represented by projections dominating $e_i^{(0)}$. Therefore we can replace $e_i^{(0)}$ by a possibly larger projection in E_i whose class in $K_0(E_i)$ is zero. The K_0-class of $e_i^{(0)}$ in B is then zero as well. Note that the equation $g_{k+3}(b(t_i))g_{k+2}(b(t_i)) = g_{k+2}(b(t_i))$ implies that $g_{k+3}(b(t_i))x = g_{k+2}(b(t_i))$ for all $x \in E_i$. In particular, with our new choice of $e_i^{(0)}$, we have this equation for $x = e_i^{(0)}$, whence

$$g_{k+3}(b(t_i)) \geq e_i^{(0)} \geq g_k(b(t_i)).$$

Similarly, there exists a projection $e_i^{(1)} \in D_{t_i}$ such that

$$g_{k+6}(b(t_i)) \geq e_i^{(1)} \geq g_{k+3}(b(t_i))$$

and $[e_i^{(1)}] = 0$ in $K_0(B)$. In particular, $e_i^{(1)} \geq e_i^{(0)}$.

Temporarily fix i, and work over the interval $[t_{i-1}, t_{i+1}]$. Observe that the element $x(t) = g_{k+3}(b(t))e_i^{(0)}g_{k+3}(b(t))$ is in the hereditary subalgebra F_t generated by $g_{k+3}(b(t))$ and satisfies $\|x(t) - e_i^{(0)}\| < 2\rho$. Since $\rho < 1/4$, functional calculus yields a projection $f_i^{(0)}(t)$, depending continuously on t, such that

$$\|f_i^{(0)}(t) - e_i^{(0)}\| < 4\rho.$$

Similarly, we obtain a projection $f_i^{(1)}(t)$ in the hereditary subalgebra generated by $g_{k+6}(b(t))$, depending continuously on $t \in [t_{i-1}, t_{i+1}]$, such that

$$\|f_i^{(1)}(t) - e_i^{(1)}\| < 4\rho.$$

We now observe that

$$\|f_i^{(1)}(t)f_i^{(0)}(t) - f_i^{(0)}(t)\| < 3(4\rho) < 14\rho,$$

one term 4ρ coming from each replacement of an $f_i^{(\nu)}(t)$ by an $e_i^{(\nu)}$. We also observe that

$$\begin{aligned}
\|e_i^{(1)}e_{i+1}^{(0)} - e_{i+1}^{(0)}\| &= \|e_i^{(1)}g_{k+3}(b)(t_{i+1})e_{i+1}^{(0)} - g_{k+3}(b)(t_{i+1})e_{i+1}^{(0)}\| \\
&\leq \|e_i^{(1)}g_{k+3}(b)(t_{i+1}) - g_{k+3}(b)(t_{i+1})\| \\
&\leq 2\|g_{k+3}(b)(t_{i+1}) - g_{k+3}(b)(t_i)\| < 2\rho,
\end{aligned}$$

since $e_i^{(1)}g_{k+3}(b)(t_i) = g_{k+3}(b)(t_i)$. An estimate similar to the one at the beginning of this paragraph therefore shows that

$$\|f_i^{(1)}(t)f_{i+1}^{(0)}(t) - f_{i+1}^{(0)}(t)\| < 14\rho$$

for $t \in [t_i, t_{i+1}]$.

Using Lemma 2.1 (3), we obtain a projection $r_i(t) \in D_t$ and a unitary path $s \mapsto v_s(t) \in \widetilde{D_t}$ for $t \in [t_{i-1}, t_i]$ and $s \in [0,1]$, both varying continuously with t, such that $r_i(t) \leq f_i^{(1)}(t)$, $v_0(t) = 1$, $v_1(t)r_i(t)v_1(t)^* = f_i^{(0)}(t)$, and

$$\|r_i(t) - f_i^{(0)}(t)\| < \beta_3(14\rho) \quad \text{and} \quad \|v_s(t) - 1\| < \beta_3(14\rho).$$

We similarly obtain $r_i'(t)$ and $v_s'(t)$ satisfying all the same conditions, except with $f_i^{(\nu)}(t)$ replaced by $f_{i-1}^{(\nu)}(t)$. We now define a continuous projection q_i on $[t_{i-1}, t_i]$ by letting $t_i' = \frac{1}{2}(t_{i-1} + t_i)$ and setting

$$q_i(t) = v_{1-\alpha}'(t)^* f_i^{(0)}(t) v_{1-\alpha}'(t) \quad \text{for } t = (1-\alpha)t_{i-1} + \alpha t_i' \text{ and } \alpha \in [0,1]$$

and

$$q_i(t) = v_\alpha(t)^* f_i^{(0)}(t) v_\alpha(t) \quad \text{for } t = (1-\alpha)t_i' + \alpha t_i \text{ and } \alpha \in [0,1].$$

This gives $q_i(t_i') = f_i^{(0)}(t_i')$ (with either definition), and

$$q_i(t_{i-1}) = r_i'(t_{i-1}) \leq f_{i-1}^{(1)}(t_{i-1}) \quad \text{and} \quad q_i(t_i) = r_i(t_i) \leq f_i^{(1)}(t_i).$$

Let $t \in [t_i', t_i]$. Then for suitable $\alpha \in [0,1]$, we have

$$\begin{aligned} \|q_i(t) - e_i^{(0)}\| &\leq \|q_i(t) - r_i(t)\| + \|r_i(t) - f_i^{(0)}(t)\| + \|f_i^{(0)}(t) - e_i^{(0)}\| \\ &< 2\|v_\alpha(t) - 1\| + \beta_3(14\rho) + 4\rho < 3\beta_3(14\rho) + 4\rho. \end{aligned}$$

A similar estimate holds for $t \in [t_{i-1}, t_i']$. Since $3\beta_3(14\rho) + 4\rho < \frac{1}{2}$, it follows that $[q_i(t)] = 0$ in $K_0(B)$, and hence also in $K_0(D_t)$, for all $t \in [t_{i-1}, t_i]$.

Since D_{t_i} is not unital, there is a projection $r_i \in D_{t_i}$ such that $r_i > f_i^{(1)}(t_i)$ and $[r_i] = 0$ in $K_0(D_{t_i})$. Lemma 2.5 provides a continuous projection $q_i' : [t_{i-1}, t_i] \to B$ such that $q_i'(t) \in D_t$, $q_i'(t_{i-1}) = r_{i-1} - q_i(t_{i-1})$, and $q_i'(t_i) = r_i - q_i(t_i)$. Now define $q(t) = q_i(t) + q_i'(t)$ for $t \in [t_{i-1}, t_i]$. Then q is well defined and continuous, since at the overlap points t_i both definitions yield r_i. Furthermore, $q \in D$ by Lemma 2.3, and $q(t) \geq q_i(t)$ whenever $t \in [t_{i-1}, t_i]$.

We now estimate, for $t \in [t_{i-1}, t_i]$:

$$\begin{aligned} &\|q_i(t)a(t) - a(t)\| \\ &\quad\leq \|q_i(t)\|\|a(t) - a(t_i)\| + \|q_i(t) - e_i^{(0)}\|\|a(t_i)\| \\ &\qquad + \|e_i^{(0)}\|\|a(t_i) - g_k(b)(t_i)a(t_i)\| + \|e_i^{(0)} g_k(b)(t_i) - g_k(b)(t_i)\|\|a(t_i)\| \\ &\qquad + \|g_k(b)(t_i)a(t_i) - a(t_i)\| + \|a(t_i) - a(t)\| \\ &\quad< \rho + (3\beta_3(14\rho) + 4\rho) + \rho + 0 + \rho + \rho \\ &\quad= 3\beta_3(14\rho) + 8\rho. \end{aligned}$$

Therefore

$$\begin{aligned} &\|q(t)a(t) - a(t)\| \\ &\quad\leq \|q(t)\|\|a(t) - q_i(t)a(t)\| + \|q(t)q_i(t) - q_i(t)\|\|a(t)\| + \|q_i(t)a(t) - a(t)\| \\ &\quad< 3(\beta_3(14\rho) + 8\rho) + 0 + (3\beta_3(14\rho) + 8\rho) = 6\beta_3(14\rho) + 16\rho \leq \varepsilon. \end{aligned}$$

A similar argument, with $p(t)$ in place of $a(t)$, shows that

$$\|q(t)p(t) - p(t)\| < \varepsilon.$$

Thus, q is the desired projection. ∎

2.7 Lemma Let B be a purely infinite simple C^*-algebra (not necessarily unital). Let $p, q \in C([0,1], B)$ be nonzero projections whose K-theory classes are equal. Then p is Murray-von Neumann equivalent to q.

Proof: Since B is purely infinite and simple, and since $p(0)$ and $q(0)$ are nonzero projections with the same class in K-theory, there is $s(0) \in B$ such that

$$s(0)s(0)^* = p(0) \quad \text{and} \quad s(0)^*s(0) = q(0).$$

Standard methods give unitary paths $t \to u(t), v(t)$ in $\tilde{B}$ such that $u(0) = v(0) = 1$ and

$$u(t)p(0)u(t)^* = p(t) \quad \text{and} \quad v(t)q(0)v(t)^* = q(t).$$

The required partial isometry is given by $s(t) = u(t)s(0)v(t)^*$. ∎

2.8 Corollary The approximate identity in Proposition 2.6 can be chosen so that the K_0-classes of all the projections in it are trivial.

Proof: Let $\{p_k\}$ be an increasing approximate identity consisting of projections. Without loss of generality, we may assume it is strictly increasing. It suffices to find projections $q_k \in D$ with $p_k \leq q_k \leq p_{k+1}$ and $[p_k] = 0$ in $K_0(D)$. Lemma 2.7 implies that both p_k and p_{k+1} are Murray-von Neumann equivalent in C([0,1], B) to the constant projections with values $p_k(0)$ and $p_{k+1}(0)$ respectively. These equivalences give an isomorphism $p_{k+1}Dp_{k+1} \to C([0,1], p_{k+1}(0)D_0p_{k+1}(0))$ (with D_0 as in Notation 2.2). Note that p_k goes to the constant projection with value $p_k(0)$. Since B is purely infinite and simple, we can choose a projection $e \in B$ such that $p_k(0) \leq e \leq p_{k+1}(0)$ and $[e] = 0$ in $K_0(B)$. Now take q_k to be the inverse image in D of $1 \otimes e$. Note that $[1 \otimes e] = 0$ in $K_0(C([0,1], p_{k+1}(0)D_0p_{k+1}(0)))$, so $[q_k] = 0$ in $K_0(D)$. ∎

2.9 Lemma Let $\psi_0, \psi_1 : \mathcal{O}_m \to B$ be two homomorphisms from an even Cuntz algebra to a purely infinite simple C^*-algebra. If $[\psi_0] = [\psi_1]$ in $KK^0(\mathcal{O}_m, B)$, and if ψ_0 and ψ_1 are both nonunital and nonzero, then ψ_0 is homotopic to ψ_1.

Proof: The condition $[\psi_0] = [\psi_1]$ in $KK^0(\mathcal{O}_m, B)$ implies that $[\psi_0(1)] = [\psi_1(1)]$ in $K_0(B)$. Therefore $\psi_0(1)$ is homotopic to $\psi_1(1)$. It follows that there is a path $t \mapsto W_t$ of unitaries in $\tilde{B}$ such that $W_0 = 1$ and $W_1^*\psi_1(1)W_1 = \psi_0(1)$. So we may assume that $\psi_0(1) = \psi_1(1)$. We can now reduce to the case that B is unital: if not, replace B by pBp for some projection $p \in B$ with $p > \psi_0(1)$.

Theorem 3.6 of [Rr1] provides a unitary $V \in \psi_0(1)B\psi_0(1)$ such that

$$\|V^*\psi_1(s_j)V - \psi_0(s_j)\| < 1/(2m)$$

for $j = 1, 2, \ldots, m$. Since B is purely infinite simple, there is a unitary $V_0 \in (1 - \psi_0(1))B(1 - \psi_0(1))$ such that $[V + V_0] = 0$ in $K_1(B)$. Replacing V by $V + V_0$, we may assume that V is a unitary in B and is in the identity component of $U(B)$. Therefore ψ_1 is homotopic to $V^*\psi_1 V$. Thus, without loss of generality, we may assume that

$$\|\psi_1(s_j) - \psi_0(s_j)\| < 1/(2m)$$

for $j = 1, 2, \ldots, m$. Then (compare [Rr1], 3.3) the unitary $U \in \psi_0(1)B\psi_0(1)$ given by

$$U = \sum_j \psi_1(s_j)\psi_0(s_j)^*$$

satisfies

$$\psi_1(s_j) = U\psi_0(s_j)$$

for all j. The formula for U implies that $\|U - 1\| < 1/2$, so there is a unitary path $t \mapsto U_t$ with $U_0 = 1$ and $U_1 = U$. Define the required homotopy by $\psi_t(s_j) = U_t\psi_0(s_j)$. ∎

2.10 Lemma Let B be purely infinite, simple, and separable. Let $t \mapsto \varphi_t$ be an asymptotic morphism from $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ to $C([0,1], B)$, with m even. Let $\psi_0, \psi_1 : C(S^1) \otimes \mathcal{O}_m \to B$ be homomorphisms. Let $M_m \subset \mathcal{O}_m$ be the $m \times m$ matrix subalgebra generated by the elements $s_i s_j^*$. Assume:

(H1) Each φ_t is linear and *-preserving, and $\sup_t \|\varphi_t\| < \infty$. (We demand neither contractivity nor positivity.)

(H2) $\varphi_t|_{C_0(S^1\setminus\{1\})\otimes M_m}$ is a homomorphism for each t.

(H3) Whenever $a \in C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ is actually in $C_0(S^1 \setminus \{1\}, p_i \mathcal{O}_m p_j)$, then $\varphi_t(a^*)\varphi_t(a)$ is in the hereditary subalgebra of $C([0,1], B)$ generated by $\varphi_t(C_0(S^1 \setminus \{1\}, \mathbf{C}p_j))$.

(H4) For $i = 0, 1$, the homomorphisms ψ_i satisfy $\mathrm{ev}_i \circ \varphi_t = \psi_i|_{C_0(S^1\setminus\{1\})\otimes\mathcal{O}_m}$ for all t.

(H5) The homomorphisms ψ_0 and ψ_1 are both nonunital, and satisfy $[\psi_0|_{\mathbf{C}\otimes\mathcal{O}_m}] = [\psi_1|_{\mathbf{C}\otimes\mathcal{O}_m}]$ in $KK^0(\mathcal{O}_m, B)$.

(H6) For all $\alpha \in [0,1]$ and $t \geq 0$, the spectrum $\mathrm{sp}(\mathrm{ev}_\alpha(\varphi_t((u-1)\otimes s_1 s_1^*)))$ is equal to the entire circle with radius 1 and center -1.

Then for all $\varepsilon > 0$ and all T, there is $t \geq T$, a unital homomorphism $\rho : \mathcal{O}_m \to C([0,1], B)$, and a unitary $v \in \rho(1)C([0,1], B)\rho(1)$ such that:

(C1) $\|v\rho(s_j) - \rho(s_j)v\| < \varepsilon$.

(C2) $\|(v - \rho(1)) - \varphi_t((u-1)\otimes 1)\| < \varepsilon$ and $\|(v - \rho(1))\rho(s_j) - \varphi_t((u-1)\otimes s_j)\| < \varepsilon$.

(C3) $\|\mathrm{ev}_i(v) - \psi_i(u \otimes 1)\| < \varepsilon$ and $\|\mathrm{ev}_i(\rho(s_j)) - \psi_i(1 \otimes s_j)\| < \varepsilon$ for $i = 0, 1$.

Lemma 2.11 below will show that an arbitrary homotopy can be modified in such a way that the hypotheses (H1)–(H3) of this lemma are satisfied. In applications, hypothesis (H6) will be achieved by forming the direct sum with a suitable homomorphism.

This lemma gives conditions under which two homomorphisms from $C(S^1) \otimes \mathcal{O}_m$ to B, with the same class in KK-theory, can be (almost) connected by a homotopy of "discrete asymptotic morphisms" (indexed by a discrete set rather than $[0, \infty)$). Presumably one can actually get a homotopy of proper asymptotic morphisms. Doing this would, however, make an already messy proof even worse, and we do not need the stronger result.

The proof of this lemma is very long and technical. The basic idea, however, is quite simple, and we therefore describe it before we begin.

For simplicity, assume in this sketch that φ is actually a homomorphism from $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ to $C([0,1]) \otimes B$. We really want a homomorphism from $C(S^1) \otimes \mathcal{O}_m$ to $C([0,1]) \otimes B$, but we will have to settle for an approximate homomorphism. The missing part is a homomorphism from $\mathcal{O}_m$ to $C([0,1]) \otimes B$. Let $T = \{\zeta - 1 : \zeta \in S^1 \setminus \{1\}\}$, so that $T \cup \{0\} = \mathrm{sp}(u-1)$. To construct our homomorphism, let $g, h \in C_c(T)$ satisfy $h(\zeta) \in T$ and $h(\zeta) \approx \zeta$ for $\zeta \in T$, $0 \leq g \leq 1$, and $gh = h$. Choose a "large" projection e_1 in the hereditary C^*-subalgebra D generated by $\varphi(h(u-1) \otimes p_1)$. (This subalgebra has an increasing approximate identity of projections by Lemma 2.5 and the hypothesis (H6).) Set $p_j = s_j s_j^* \in \mathcal{O}_m$. Note that the element $\varphi(g(u-1) \otimes p_j)$ acts as an identity for D. Thus, we can define projections

$$e_j = \varphi(g(u-1) \otimes s_j s_1^*) e_1 \varphi(g(u-1) \otimes s_1 s_j^*) \quad \text{and} \quad e = \sum_{j=1}^m e_j,$$

and partial isometries

$$t_j = e_j \varphi(g(u-1) \otimes s_j).$$

Now $h(u-1)$ is close to $u-1$, and e_1 is fairly far out in an approximate identity for an algebra which contains $\varphi(h(u-1)\otimes p_1)$. Therefore the elements we have constructed approximately commute with $\varphi((u-1)\otimes 1)+1$. Furthermore, it is quite easy to extend the t_j to have larger initial and final projections and still approximately commute with $\varphi((u-1) \otimes 1) + 1$.

Unfortunately, there is a problem. We certainly have $t_j t_j^* = e_j$, but there is no reason to have $t_j^* t_j = e$. The best we can say is that $t_j^* t_j$ and e are close, not in norm but only in the strict topology. (Both are "large" in

a certain hereditary subalgebra.) In order to get objects satisfying the relations for $\mathcal{O}_m$, we need to correct for this error. To have room for the correction, we must repeat the argument of the previous paragraph using a second pair of functions $g', h' \in C_c(T)$. These functions are to satisfy the same relations as g and h, and they are supposed to be "larger". The precise condition is $h'(\zeta) = \zeta$ on the support of g. Let e'_j and $e' = \sum_j e'_j$ be the new, larger, projections. It is crucial for the proof that e' strictly dominates both e and the common initial projection $t_j^* t_j$ of the original set of partial isometries.

The correction procedure makes the proof much more complicated. (Among other problems, we can't exactly get the domination referred to at the end of the previous paragraph. We must settle for an ε approximation.) Even worse, we can't work with a homomorphism φ, but only with an approximate homomorphism φ_t taken from the given asymptotic morphism. As a result, errors accumulate with every step in the construction. The proof ends with a device to get things to match up at the endpoints of the homotopy, which we have not discussed in this sketch.

Proof of Lemma 2.10: We divide the proof into nine steps, of which Steps 1 and 6 are further subdivided.

Step 1: Preliminaries. The purpose of this step is to set things up for the rest of the proof.

Step 1.1: We begin by making several reductions and definitions.

We usually write α for an element of $[0, 1]$, and use function notation for the dependence of an element of $C([0, 1], B)$ on α. We use subscripts for the parameter t of the asymptotic morphism.

For simplicity, we assume that B is unital.

Since the conclusion of the lemma only involves approximations to within ε, we can make a preliminary small perturbation of $t \to \varphi_t$. By making such a perturbation, we may assume that $\mathrm{ev}_\alpha \circ \varphi_t$ is a constant function of α for α in some neighborhood of 0 and also for α in some neighborhood of 1. We now reparametrize the interval $[0, 1]$ so as to be able to assume that these neighborhoods are $[0, 1/3]$ and $[2/3, 1]$, respectively.

Set $p_j = s_j s_j^* \in \mathcal{O}_m$.

Step 1.2: We now choose some small numbers η, δ, δ', and δ'', some useful functions defined on a circle in $\mathbf{C}$, and a suitable large value t of the parameter in our asymptotic morphism.

Using the stability of the defining relations for $\mathcal{O}_m$ (see the proof of Lemma 2.1 (5)), choose $\eta > 0$ such that whenever A is a C^*-algebra and $\sigma_0, \sigma_1 : \mathcal{O}_m \to A$ are homomorphisms such that $\|\sigma_0(s_j) - \sigma_1(s_j)\| < \eta$ for $j = 1, \ldots, m$, then σ_0 is homotopic to σ_1.

Let $M_0 = \sup\{\|\varphi_t\| : t \in [0, 1]\}$. Note that $M_0 \geq 1$. For $\delta > 0$, define, using the functions $\beta_1, \beta_2, \ldots$ from Lemma 2.1:

$$\begin{aligned} M &= M_0 + \delta, \\ \delta' &= \left[M^2 \beta_3(\beta_2(\delta) + \delta) + (M^2 + 1)\beta_1(M^2\delta) + 2(M^2 + 1)\delta\right]^{1/2}, \end{aligned} \tag{1}$$

and

$$\delta'' = \max\left(2\delta,\ \beta_2\left(\beta_2(\delta') + (M^2 + 2)\beta_1(M^2\delta) + 2M^2\delta\right) + \beta_1(M^2\delta)\right). \tag{2}$$

Now choose $\delta > 0$ so that:

$$\beta_2(2\beta_2(\delta)) + \beta_2(\delta) < 1/2, \tag{3}$$

$$\beta_4\left(\delta'' + \beta_2(\delta') + \beta_1(M^2\delta)\right) < \infty, \tag{4}$$

$$4M^3(m + 1)\beta_4(\delta'') + 2\beta_4(2M^3\delta) + 10M^3(m + 1)\delta < \varepsilon, \tag{5}$$

and

$$\beta_5(\delta + \beta_4(\delta'')) < \min(\eta, \varepsilon). \tag{6}$$

Define $T = \{\zeta - 1 : \zeta \in S^1 - \{1\}\}$. (Thus, if v is unitary, then $\mathrm{sp}(v-1) \subset T \cup \{0\}$.) Choose functions $g, g', h, h' \in C_c(T)$ satisfying the following properties:

$$\begin{aligned}&0 \le g, g' \le 1, \quad g'g = g,\\&h(T), h'(T) \subset T \cup \{0\} \quad \text{and} \quad |h(\zeta) - \zeta|, |h'(\zeta) - \zeta| < \delta,\\&gh = h, \quad g'h' = h', \quad \text{and} \quad h'(\zeta) = \zeta \text{ on } \mathrm{supp}(g).\end{aligned}$$

Note that g' and h' are not the derivatives of g and h.

Choose, and fix, t so large that the estimates

$$\left\| \varphi_t\left(\prod_{i=1}^{l} x_i\right) - \prod_{i=1}^{l} \varphi_t(x_i) \right\| < \delta \tag{7}$$

hold for $1 \le l \le 7$ and $x_1, \ldots, x_l$ of the form $a \otimes b$, with a or a^* in

$$\{g(u-1),\, g(u-1)^{1/3},\, g'(u-1),\, g'(u-1)^{1/3},\, h(u-1),\, h'(u-1)\}$$

and with b or b^* in $\{1, s_j, p_j\}$. (To get such an estimate for products of three or more elements, we use $\|\varphi_t\| \le M_0$ for all t. Note that our hypotheses imply $\varphi_t(x)\varphi_t(y) = \varphi_t(xy)$ for x, y both of the form $a \otimes b$, with a as above and $b \in \{1, p_j\}$.)

Step 1.3: We now choose elements of $C([0,1], B)$ to be used to build the partial isometries and matrix units that we will need.

Define the following elements of $C([0,1], B)$:

$$\begin{aligned}a_j &= \varphi_t(h(u-1) \otimes p_j),\\w_j &= \varphi_t(g(u-1)^{1/3} \otimes p_j)\varphi_t(g(u-1)^{1/3} \otimes s_j)\varphi_t(g(u-1)^{1/3} \otimes 1),\\\tilde{w}_{ij} &= \varphi_t(g(u-1) \otimes s_i s_j^*).\end{aligned}$$

Further define a_j', w_j', and $\tilde{w}_{ij}'$ in the same way, but using $g'(u-1)$ and $h'(u-1)$ in place of $g(u-1)$ and $h(u-1)$. The first and last factors in the definition of w_j and w_j' ensure that, for example, $w_j w_j^*$ is in the hereditary subalgebra generated by $\varphi_t(g(u-1)) \otimes p_j)$. The powers of $g(u-1)$ and $g'(u-1)$ are chosen to as to have $\tilde{w}_{ij}$ close to $w_i w_j^*$ and $\tilde{w}_{ij}'$ close to $w_i'(w_j')^*$.

The estimate (7) implies relations of the form

$$\begin{aligned}&\|a_j w_j - w_j\left(\sum_{i=1}^{m} a_i\right)\| < 2\delta,\\&\|\tilde{w}_{ij}' w_j - w_i\| < 2\delta, \quad \|w_i w_j^* - \tilde{w}_{ik}\tilde{w}_{kj}^*\| < 2\delta, \quad \text{etc.},\end{aligned} \tag{8}$$

since both terms in each difference differ by less than δ from $\varphi(x)$ for some x. Among $\tilde{w}_{ij}, \tilde{w}_{ij}'$, a_j, and a_j', one actually gets equalities at these places, for example:

$$a_i \tilde{w}_{ij} = \tilde{w}_{ij} a_j, \quad \tilde{w}_{ij}\tilde{w}_{kl} = \delta_{jk}\tilde{w}_{ii}\tilde{w}_{il} \quad \text{etc.} \tag{9}$$

Also note that

$$\tilde{w}_{ji} = \tilde{w}_{ij}^* \quad \text{and} \quad \tilde{w}_{ji}' = (\tilde{w}_{ij}')^*.$$

Since $\|\varphi_t\| \le M_0$, we have $\|a_j\| \le 2M_0 \le 2M$ for each j, and all the other elements listed above have norm at most either $M_0 \le M$ or $M_0 + \delta = M$.

Step 1.4: We define two important hereditary subalgebras D and D', which will play a crucial role in the proof.

Let D be the hereditary C^*-subalgebra of $C([0,1],B)$ generated by a_1. Then for each $\alpha \in [0,1]$, the subalgebra $D_\alpha \subset B$ (defined as in Notation 2.2) is the hereditary C^*-subalgebra generated by

$$\mathrm{ev}_\alpha(a_1) = \mathrm{ev}_\alpha(\varphi_t(h(u-1) \otimes p_1)) = h(\mathrm{ev}_\alpha \circ \varphi_t((u-1) \otimes p_1)).$$

(This uses the fact that $\varphi_t|_{C_0(S^1-\{1\}) \otimes M_m}$ is a homomorphism.) Since $\mathrm{ev}_\alpha \circ \varphi_t((u-1) \otimes p_1)$ has full spectrum by assumption (full here means equal to $T \cup \{0\} = \{\zeta - 1 : \zeta \in S^1\}$, the largest it can be), and since the range of h is $T \cup \{0\}$, the element $\mathrm{ev}_\alpha(a_1)$ also has full spectrum. It follows that each D_α is nonzero and nonunital. Therefore Proposition 2.6 and Corollary 2.8 imply that D has an increasing approximate identity consisting of projections whose classes in $K_0(D)$ are trivial.

The same reasoning applies to the hereditary subalgebra D' generated by a_1'. Note that $D \subset D'$.

Since a_1 and a_1' are normal, we have

$$D = \overline{a_1 C([0,1],B) a_1^*} = \overline{a_1^* C([0,1],B) a_1}$$

and similarly for a_1' and D'.

We observe the following important property of the hereditary subalgebras D and D' :

(M) If $x \in D$, then $\tilde{w}_{11} x = x \tilde{w}_{11} = x$.

(M$'$) If $x \in D'$, then $\tilde{w}_{11}' x = x \tilde{w}_{11}' = x$.

Property (M) follows by a standard argument from the relations $\tilde{w}_{11} a_1 = \tilde{w}_{11} a_1 = a_1$ and $\tilde{w}_{11} a_1^* = a_1^* \tilde{w}_{11} = a_1^*$. Property (M$'$) is similar. Similar arguments also give the following properties:

(Z) If $x \in D$, then $\tilde{w}_{ij} x = x \tilde{w}_{ji} = x w_j = 0$ for arbitrary i and for $j \neq 1$.

(Z$'$) If $x \in D'$, then $\tilde{w}_{ij}' x = x \tilde{w}_{ji}' = x w_j' = 0$ for arbitrary i and for $j \neq 1$.

Step 2: We now construct a first (lower) level of matrix units $\{e_{ij}\}$ in D. These look like they should span the copy of M_m inside a homomorphic image of $\mathcal{O}_m$, but unfortunately things are not that easy.

Choose a projection $e_1 \in D$ (from the approximate identity obtained above) such that $[e_1] = 0$ in $K_0(D)$ and

$$\|e_1 a_1 - a_1\|, \; \|a_1 e_1 - a_1\| < \delta. \tag{10}$$

Define

$$e_{ij} = \tilde{w}_{i1} e_1 \tilde{w}_{1j}, \quad e_j = e_{jj}, \quad \text{and} \quad e = \sum_{j=1}^{m} e_j.$$

We claim that $(e_{ij})_{i,j=1}^m$ is a system of matrix units in $C([0,1],B)$. To prove this, we first note that clearly $e_{ij}^* = e_{ji}$. Furthermore, using the properties (M) and (Z) from Step 1.4,

$$e_{ij} e_{kl} = \tilde{w}_{i1} e_1 \tilde{w}_{1j} \tilde{w}_{k1} e_1 \tilde{w}_{1l} = \delta_{jk} \tilde{w}_{i1} e_1 \tilde{w}_{11}^2 e_1 \tilde{w}_{1l} = \delta_{jk} e_{il}.$$

Step 3: We now construct a second level of matrix units $\{e_{ij}'\}$ in D'. They need to be enough bigger than the first ones to allow room for a correction for the failure of $w_j^* e_j w_j$ to be close to e. To ensure this, we first construct projections r, f, and e_1' in D' such that $\|f - w_1 e w_1^*\|$ is small, $e_1, f \leq r$, and $r < e_1'$. We will need r in the next step, but all the other intermediate projections constructed in this step can be discarded after it is done.

The definition of w_1 implies that $w_1 e w_1^* \in D'$. Furthermore, we have the following estimate, in which the second step uses properties (M) and (Z), and the last step is similar to (8):

$$\begin{aligned} \|(w_1 e w_1^*)^2 - w_1 e w_1^*\| &\leq M^2 \|e w_1^* w_1 e - e\| \\ &= M^2 \left\| e w_1^* w_1 e - e \left(\sum_j \tilde{w}_{jj}^2 \right) e \right\| \leq M^2 \left\| w_1^* w_1 - \left(\sum_j \tilde{w}_{jj}^2 \right) \right\| < M^2 \delta. \end{aligned}$$

Therefore there exists a projection $f_0 \in D'$ such that

$$\|f_0 - w_1 e w_1^*\| < \beta_1(M^2\delta).$$

Since D' has an increasing approximate identity of projections with trivial K_0-classes, we can choose a projection $r_0 \in D'$ such that $[r_0] = 0$ in $K_0(D')$ and

$$\|r_0 e_1 - e_1\| < \delta \quad \text{and} \quad \|r_0 f_0 - f_0\| < \delta, \tag{11}$$

and then choose a projection $r_1 \in D'$ such that $r_1 > r_0$ and $[r_1] = 0$ in $K_0(D')$. It follows that there is a projection $r \in D'$ such that $r \geq e_1$ and

$$\|r - r_0\| < \beta_2(\delta). \tag{12}$$

We then have $\|rf_0 - f_0\| < \beta_2(\delta) + \delta$. Therefore there is a projection $f \leq r$ such that $\|f_0 - f\| < \beta_3(\beta_2(\delta) + \delta)$, and it follows that

$$\|f - w_1 e w_1^*\| < \beta_1(M^2\delta) + \beta_3(\beta_2(\delta) + \delta). \tag{13}$$

We also have $\|r_1 r - r\| \leq 2\|r - r_0\| < 2\beta_2(\delta)$. Using Lemma 2.1 (2) and combining the resulting estimates with ones we already have (including (3)), we obtain a projection $e_1' \in D'$ such that $e_1' \geq r$ and

$$\|(e_1' - r) - (r_1 - r_0)\| < \beta_2(2\beta_2(\delta)) + \beta_2(\delta) < 1/2.$$

Since $r_1 > r_0$, it follows that $e_1' > r$. Similar arguments show that the classes of f, r, and e_1' in $K_0(D')$ are all zero.

We now define

$$e_{ij}' = \tilde{w}_{i1}' e_1' \tilde{w}_{1j}', \quad e_j' = e_{jj}', \quad \text{and} \quad e' = \sum_j e_j'.$$

Then $\{e_{ij}'\}$ is a system of matrix units by the same argument as for $\{e_{ij}\}$. (See Step 2.) We have $e_1' \geq e_1$ by construction, so

$$e_j e_j' = \tilde{w}_{j1} e_1 \tilde{w}_{1j} \tilde{w}_{j1}' e_1' \tilde{w}_{1j}' = \tilde{w}_{j1} e_1 e_1' \tilde{w}_{1j}' = e_j,$$

that is, $e_j \geq e_j'$. It follows that $e' \geq e$. Since we actually have $e_1' > e_1$, we in fact get $e' > e$. Furthermore, similar arguments show

$$e_i e_{ij}' = e_{ij} = e_{ij}' e_j.$$

Step 4: We now construct a projection $p > e$ such that $[p] = 0$ in $K_0(C([0,1], B))$ and $\|p - (w_1')^* e_1' w_1'\|$ is small. In order to ensure that $p > e$, we use an intermediate projection $p_0 \geq e$ such that $\|p_0 - (w_1')^* r w_1'\|$ is small.

We estimate, using reasoning similar to that for (8):

$$\begin{aligned}
&\|((w_1')^* r w_1')^2 - (w_1')^* r w_1'\| \leq M^2 \|r w_1' (w_1')^* r - r\| \\
&\quad = M^2 \|r w_1' (w_1')^* r - r (\tilde{w}_{11}')^2 r\| \leq M^2 \|w_1' (w_1')^* - (\tilde{w}_{11}')^2\| < M^2\delta.
\end{aligned}$$

Therefore there is a projection p_1 such that

$$\|p_1 - (w_1')^* r w_1'\| < \beta_1(M^2\delta).$$

Similarly, there is a projection p_2 such that

$$\|p_2 - (w_1')^* e_1' w_1'\| < \beta_1(M^2\delta).$$

Using $f \leq r$ in the third step and (13) in the fourth step, we estimate:

$$\begin{aligned}
\|p_1 e - e\|^2 &= \|e(1 - p_1)e\| \leq \|p_1 - (w_1')^* r w_1'\| + \|e - e(w_1')^* r w_1' e\| \\
&< (1 + M^2)\beta_1(M^2\delta) + \|e(1 - (w_1')^* f w_1')e\| \\
&< (1 + M^2)\beta_1(M^2\delta) + M^2\beta_3(\beta_2(\delta) + \delta) + \|e(1 - (w_1')^* w_1 e w_1^* w_1')e\|
\end{aligned} \tag{14}$$

To estimate the last term, we first observe that

$$e \sum_j \varphi_t(g(u-1)^{1/3} \otimes p_j) = \sum_j e_j(\tilde{w}_{jj})^{1/3} = e = e \sum_j \varphi_t(g'(u-1)^{1/3} \otimes p_j).$$

Therefore $e w_1^* = e(w_1')^*$. So we can replace w_1 by w_1' in the last term of (14). Now

$$\begin{aligned} \|e(w_1')^* w_1' - e\| &= \|e \left(\sum_j \tilde{w}_{jj} \right) (w_1')^* w_1' - e \left(\sum_j \tilde{w}_{jj} \right) \| \\ &\leq \|e\| \| \left(\sum_j \tilde{w}_{jj} \right) (w_1')^* w_1' - \left(\sum_j \tilde{w}_{jj} \right) \|. \end{aligned}$$

We have $\sum_j \tilde{w}_{jj} = \varphi_t(g_1(u-1) \otimes 1)$. Furthermore, $\left(\sum_j \tilde{w}_{jj}\right)(w_1')^* w_1'$ is the product of 7 factors $\varphi_t(x_1) \cdots \varphi_t(x_7)$, with the x_i in the list of elements to which (7) applies, and $x_1 \cdots x_7 = g_1(u-1) \otimes 1$. Therefore

$$\|e(w_1')^* w_1' - e\| \leq 2\delta.$$

Applying this inequality and its adjoint, we can replace the middle factor in the last term of (14) by $1-e$ at a cost of $2(M^2+1)\delta$. Therefore, using (1), we get

$$\|p_1 e - p_1\| < [M^2 \beta_3(\beta_2(\delta) + \delta) + (M^2+1)\beta_1(M^2\delta) + 2(M^2+1)\delta]^{1/2} = \delta'.$$

It follows that there is a projection $p_0 \geq e$ such that $\|p_0 - p_1\| < \beta_2(\delta')$, whence

$$\|p_0 - (w_1')^* r w_1'\| < \beta_2(\delta') + \beta_1(M^2\delta). \tag{15}$$

We now have

$$\begin{aligned} &\|p_2 p_0 - p_0\| \leq 2\|p_0 - (w_1')^* r w_1'\| \\ &\quad + \|(w_1')^* r w_1'\| \|p_2 - (w_1')^* e_1' w_1'\| + \|(w_1')^* e_1' w_1' (w_1')^* r w_1' - (w_1')^* r w_1'\|. \end{aligned}$$

The reasoning of (8) shows that $\|w_1'(w_1')^* - (\tilde{w}_{11}')^2\| < 2\delta$; furthermore, $e_1'(\tilde{w}_{11}')^2 r = e_1' r = r$. So the last term above has norm at most $2M^2\delta$, and we get

$$\|p_2 p_0 - p_0\| < 2(\beta_2(\delta') + \beta_1(M^2\delta)) + M^2\beta_1(M^2\delta) + 2M^2\delta.$$

Applying Lemma 2.1 (2) in the usual way, we find a projection $p \geq p_0$ such that

$$\begin{aligned} &\|p - (w_1')^* e_1' w_1'\| \leq \|p - p_2\| + \|p_2 - (w_1')^* e_1' w_1'\| \\ &\quad < \beta_2\left(\beta_2(\delta') + (2+M^2)\beta_1(M^2\delta) + 2M^2\delta\right) + \beta_1(M^2\delta) \leq \delta''. \end{aligned} \tag{16}$$

(See (2) for the definition of δ''.) We note that

$$\|(p - p_0') - (w_1')^*(e_1' - r) w_1'\| < \delta'' + \beta_2(\delta') + \beta_1(M^2\delta)$$

by (15) and (16), and $\|w_1'(w_1')^* - (\tilde{w}_{11}')^2\| < 2\delta$ implies (using (2))

$$\|(e_1' - r) w_1' (w_1')^* (e_1' - r) - (e_1' - r)\| < 2\delta \leq \delta'' + \beta_2(\delta') + \beta_1(M^2\delta).$$

Since $\beta_4(\delta'' + \beta_2(\delta') + \beta_1(M^2\delta)) < \infty$ by (4), Lemma 2.1 (4) implies that $p - p_0'$ is Murray-von Neumann equivalent to $e_1' - r$, and therefore nonzero. Similar estimates show p is Murray-von Neumann equivalent to e_1' and p_0' is Murray-von Neumann equivalent to r. It follows that $[p] = [p_0'] = 0$ in $K_0(C([0,1], B))$.

Step 5: We now construct a homomorphism $\rho_0 : \mathcal{O}_m \to C([0,1]) \otimes B$. This homomorphism will be one direct summand in the homomorphism we need for the proof of the lemma.

Since $p > p_0'$ and $p_0' \geq e$, we get $p > e$; also recall from above that $e' > e$. Since all K_0-classes are trivial, Lemma 2.7 implies that $p - e$ and $e' - e$ are Murray-von Neumann equivalent. Thus, there exists a partial isometry $d \in C([0,1]) \otimes B$ such that

$$d^* d = e', \quad dd^* = p, \quad \text{and} \quad de = ed = e.$$

Furthermore, the estimates

$$\|p - (w_1')^* e_1' w_1'\| < \delta'' \quad \text{and} \quad \|e_1' w_1' (w_1')^* e_1' - e_1'\| < 2\delta \leq \delta''$$

imply the existence of a partial isometry $\overline{w}_1$ such that

$$\overline{w}_1 \overline{w}_1^* = e_1', \quad \overline{w}_1^* \overline{w}_1 = p, \quad \text{and} \quad \|e_1' w_1' - \overline{w}_1\| < \beta_4(\delta''). \tag{17}$$

Note that

$$(\overline{w}_1 d)^* (\overline{w}_1 d) = e' \quad \text{and} \quad (\overline{w}_1 d)(\overline{w}_1 d)^* = e_1'.$$

Define

$$z_j = e_{j1}' \overline{w}_1 d.$$

Then

$$z_j^* z_j = d^* \overline{w}_1^* e_{1j}' e_{j1}' \overline{w}_1 d = d^* \overline{w}_1^* e_1' \overline{w}_1 d = e'$$

and

$$z_j z_j^* = e_{j1}' \overline{w}_1 d d^* \overline{w}_1^* e_{1j}' = e_{j1}' e_1' e_{1j}' = e_j'.$$

Thus, there exists a homomorphism $\rho_0 : \mathcal{O}_m \to C([0,1], B)$ such that $\rho_0(s_j) = z_j$ and $\rho_0(1) = e'$.

Step 6: Next, we construct a unitary $v_0 \in e' C([0,1], B) e'$ which approximately commutes with the range of ρ_0. For later use, we actually do the following more general calculation. Let $\tilde{d}$ be a partial isometry and let $\tilde{e}'$ be a projection such that:

$$\tilde{d}\tilde{d}^* \geq p, \quad \tilde{d}e = e\tilde{d} = e, \quad \tilde{e}' \geq \tilde{d}^* \tilde{d}, \quad \text{and} \quad \tilde{e}' \geq e'. \tag{18}$$

Set

$$\tilde{c} = \tilde{e}' + \sum_{j=1}^{m} e_j a_j e_j. \tag{19}$$

Define $\tilde{z}_j = e_{j1}' \overline{w}_1 \tilde{d}$. (That is, use $\tilde{d}$ in place of d in the definition of z_j.) Then $\tilde{c}$ is close to a unitary $\tilde{v}_0 = \tilde{c}(\tilde{c}^* \tilde{c})^{-1/2} \in \tilde{e}' C([0,1], B) \tilde{e}'$ which approximately commutes with $\tilde{z}_j$ for all j.

For the special case $\tilde{d} = d$ of immediate interest, we set $\tilde{c} = c$ and $\tilde{v}_0 = v_0$.

A number of the intermediate estimates will be needed in Step 7.

Step 6.1: We show that $\tilde{c}$ is approximately unitary.

First, observe that

$$\begin{aligned} \|a_j e_j - a_j\| &= \|\tilde{w}_{j1} a_1 \tilde{w}_{1j} e_j - \tilde{w}_{j1} a_1 \tilde{w}_{1j}\| = \|\tilde{w}_{j1} a_1 e_1 \tilde{w}_{1j} - \tilde{w}_{j1} a_1 \tilde{w}_{1j}\| \\ &\leq M^2 \|a_1 e_1 - a_1\| < M^2 \delta. \end{aligned}$$

Since $\|a_j^*\| \leq 2M$, we get $\|a_j^* e_j a_j - a_j^* a_j\| < 2M^3 \delta$. Combining this with the fact that the e_j are orthogonal projections dominated by $\tilde{e}'$ and the fact (similar to (9)) that $a_j^* a_j - a_j - a_j^* = 0$, we obtain:

$$\begin{aligned} \|\tilde{c}^* \tilde{c} - \tilde{e}'\| &= \|\sum_j e_j (a_j^* e_j a_j - a_j - a_j^*) e_j\| \\ &\leq \max_j \|a_j^* e_j a_j - a_j - a_j^*\| < 2M^3 \delta. \end{aligned} \tag{20}$$

A similar estimate shows that
$$\|\tilde{c}\tilde{c}^* - \tilde{e}'\| < 2M^3\delta. \tag{21}$$

Step 6.2: We prove that $\tilde{c}$ commutes with e'_{j1}, the first factor in the definition of $\tilde{z}_j$.

First, certainly $\tilde{e}' \geq e' = \sum_i e'_{ii}$ commutes with e'_{j1}.

For the other part, we start with the equation $\tilde{w}_{1j}a_j\tilde{w}_{j1} = \tilde{w}_{11}a_1\tilde{w}_{11}$, which follows from the same reasoning as (9). Therefore

$$e_ja_je_{j1} = (\tilde{w}_{j1}e_1\tilde{w}_{1j})a_j(\tilde{w}_{j1}e_1\tilde{w}_{11}) = (\tilde{w}_{j1}e_1\tilde{w}_{11})a_1(\tilde{w}_{11}e_1\tilde{w}_{11}) = e_{j1}a_1e_1.$$

Now one checks that

$$\left(\sum_{i=1}^m e_ia_ie_i\right)e'_{j1} = e_ja_je_{j1} \quad \text{and} \quad e'_{j1}\left(\sum_{i=1}^m e_ia_ie_i\right) = e_{j1}a_1e_1.$$

So $\tilde{c}$ commutes with e'_{j1}.

Step 6.3: We next show that $\tilde{c}$ approximately commutes with $\overline{w}_1\tilde{d}$, the other factor in the definition of $\tilde{z}_j$.

Again, $\tilde{e}'$ is easy. We have $\tilde{e}' \geq e' \geq e'_1$, so $\tilde{e}'\overline{w}_1\tilde{d} = \overline{w}_1\tilde{d}$, and also $\tilde{e}' \geq \tilde{d}^*\tilde{d}$, so $\overline{w}_1\tilde{d}\tilde{e}' = \overline{w}_1\tilde{d}$. So we get exact commutativity here:
$$\tilde{e}'\overline{w}_1\tilde{d} - \overline{w}_1\tilde{d}\tilde{e}' = 0. \tag{22}$$

The other part is longer. We begin by using the relations $e_j\overline{w}_1 = e_je'_1\overline{w}_1 = \delta_{1j}e_1\overline{w}_1$ and $\tilde{d}e_j = e_j\tilde{d} = e_j$, to get

$$\begin{aligned}\tilde{c}\overline{w}_1\tilde{d} - \overline{w}_1\tilde{d}\tilde{c} &= \left(\sum_{j=1}^m e_ja_je_j\right)\overline{w}_1\tilde{d} - \overline{w}_1\tilde{d}\left(\sum_{j=1}^m e_ja_je_j\right) \\ &= e_1a_1e_1\overline{w}_1\tilde{d} - \overline{w}_1\left(\sum_{j=1}^m e_ja_je_j\right)\tilde{d}.\end{aligned}$$

Using $\|\overline{w}_1 - e'_1w'_1\| < \beta_4(\delta'')$, $e_1e'_1 = e_1$, and $\|a_j\| \leq 2M$, we get

$$\begin{aligned}\|\tilde{c}\overline{w}_1\tilde{d} - \overline{w}_1\tilde{d}\tilde{c}\| &< 2M(m+1)\beta_4(\delta'') \\ &\quad + \|e_1a_1e_1w'_1\tilde{d} - e'_1w'_1\left(\sum_{j=1}^m e_ja_je_j\right)\tilde{d}\|.\end{aligned} \tag{23}$$

Now (10) implies $\|e_1a_1e_1 - a_1\| < 2\delta$, and

$$e_ja_je_j - a_j = (\tilde{w}_{j1}e_1\tilde{w}_{1j})a_j(\tilde{w}_{j1}e_1\tilde{w}_{1j}) - \tilde{w}_{j1}a_1\tilde{w}_{1j} = \tilde{w}_{j1}(e_1a_1e_1 - a_1)\tilde{w}_{1j},$$

so
$$\|e_ja_je_j - a_j\| < 2M^2\delta. \tag{24}$$

We apply this to each of the $m+1$ a's in (23), and use $\|\tilde{d}\| \leq 1$ and $\|w'_1\| \leq M$, to get:

$$\begin{aligned}\|\tilde{c}\overline{w}_1\tilde{d} - \overline{w}_1\tilde{d}\tilde{c}\| &< 2M(m+1)\beta_4(\delta'') + 2M^3(m+1)\delta \\ &\quad + \|a_1 - e'_1a_1\|\|w'_1\| + \|e'_1\|\|a_1w'_1 - w'_1\sum_j a_j\|.\end{aligned}$$

Now

$$\|a_1 w_1' - w_1' \sum_j a_j\| = \|a_1 w_1' - \sum_j w_1' a_j\| < (m+1)\delta,$$

by reasoning similar to that which gave (8). (One needs to use (7) on products of four factors.) Furthermore,

$$\|a_1 - e_1' a_1\| = \|a_1(1-e_1')a_1\|^{1/2} \leq \|a_1(1-e_1)a_1\|^{1/2} < \delta, \tag{25}$$

by (10). So

$$\begin{aligned} \|\tilde{c}\overline{w}_1\tilde{d} - \overline{w}_1\tilde{d}\tilde{c}\| &< 2M(m+1)\beta_4(\delta'') + 2M^3(m+1)\delta + M\delta + (m+1)\delta \\ &\leq 2M^3(m+1)(\beta_4(\delta'') + 2\delta). \end{aligned} \tag{26}$$

Step 6.4: We now combine our estimates to get an estimate on the commutator of $\tilde{z}_j$ with the unitary $\tilde{c}(\tilde{c}^*\tilde{c})^{-1/2}$.

Lemma 2.1 (4) and the estimates (20) and (21) show that the unitary $\tilde{v}_0 = \tilde{c}(\tilde{c}^*\tilde{c})^{-1/2} \in \tilde{e}' C([0,1], B)\tilde{e}'$ satisfies

$$\|\tilde{v}_0 - \tilde{c}\| < \beta_4(2M^3\delta). \tag{27}$$

It is also important to notice that

$$\tilde{v}_0(\tilde{e}' - e) = (\tilde{e}' - e)\tilde{v}_0 = \tilde{e}' - e. \tag{28}$$

This follows from the analogous fact for $\tilde{c}$.

From (22), (26), and the definition of $\tilde{z}_j$, we get

$$\|\tilde{c}\tilde{z}_j - \tilde{z}_j\tilde{c}\| < 2M^3(m+1)(\beta_4(\delta'') + 2\delta). \tag{29}$$

Therefore

$$\|\tilde{v}_0\tilde{z}_j - \tilde{z}_j\tilde{v}_0\| < 2M^3(m+1)(\beta_4(\delta'') + 2\delta) + 2\beta_4(2M^3\delta) < \varepsilon. \tag{30}$$

(The last step is a weaker relation than (5).)

Step 7: Continuing with the hypotheses and notation of Step 6, we verify the following estimates, which are analogous to those in the conclusion (C2):

$$\|(\tilde{v}_0 - \tilde{e}') - \varphi_t((u-1)\otimes 1)\|, \ \|(\tilde{v}_0 - \tilde{e}')\tilde{z}_j - \varphi_t((u-1)\otimes s_j)\| < \varepsilon.$$

For the first, we observe

$$\begin{aligned} &\|\tilde{v}_0 - \tilde{e}' - \varphi((u-1)\otimes 1)\| \\ &\quad\leq \|\tilde{v}_0 - \tilde{c}\| + \|\tilde{c} - \tilde{e}' - \sum_j a_j\| + \|\varphi_t(h(u-1)\otimes 1) - \varphi_t((u-1)\otimes 1)\| \\ &\quad< \beta_4(2M^3\delta) + \sum_j \|e_j a_j e_j - a_j\| + \|\varphi_t\|\|h(u-1) - (u-1)\| \\ &\quad< \beta_4(2M^3\delta) + 2M^2 m\delta + M\delta < \varepsilon. \end{aligned}$$

(The last step is a weaker relation than (5).)

For the second estimate, we do the following approximations, in which we give the justifications and errors afterwards:

$$\begin{aligned} (\tilde{v}_0 - \tilde{e}')\tilde{z}_j &\approx (\tilde{c} - \tilde{e}')\tilde{z}_j \approx \tilde{z}_j(\tilde{c} - \tilde{e}') = e_{j1}'\overline{w}_1\tilde{d}\sum_i e_i a_i e_i \\ &\approx e_{j1}' w_1' \sum_i e_i a_i e_i \approx e_{j1}' w_1' \sum_i a_i \\ &= \tilde{w}_{j1}' e_1' \tilde{w}_{11}' w_1' \sum_i a_i \approx \tilde{w}_{j1}' e_1' a_1 \tilde{w}_{11}' w_1' \approx \tilde{w}_{j1}' a_1 \tilde{w}_{11}' w_1' \\ &\approx \varphi_t(h(u-1)\otimes s_j) \approx \varphi_t((u-1)\otimes s_j). \end{aligned}$$

The errors are, in order (including the equality signs), $\beta_4(2M^3\delta)$ (by (27)), $2M^3(m+1)(\beta_4(\delta'')+2\delta)$ (by (29)), 0 (by the definitions of $\tilde{z}_j$ and $\tilde{c}$),
$2Mm\beta_4(\delta'')$ (by (17) and because $\tilde{d}e_i = e_i$ and $e'_{j1}e'_1 = e'_{j1}$), $Mm(2M^2\delta)$ (by (24)), 0, $M(m+1)\delta$ (by reasoning similar to (8), using (7) on products of 5 factors), $M^3\delta$ (by (25)), δ (by (7), applied to a product of 6 factors), and $M\delta$ (since $\|\varphi_t\| \leq M$ and by the choice of h). Adding up the errors, and replacing the result by a larger but simpler expression, we find that

$$\begin{aligned} &\|(\tilde{v}_0 - \tilde{e}')\tilde{z}_j - \varphi_t((u-1)\otimes s_j)\| \\ &\qquad < 4M^3(m+1)\beta_4(\delta'') + \beta_4(2M^3\delta) + 10M^3(m+1)\delta < \varepsilon, \end{aligned} \tag{31}$$

by (5).

Step 8: In this step, we fill out ρ_0 and v_0 to produce the homomorphism and unitary we actually need. Because we have worked in a "large" hereditary subalgebra, we can extend v_0 by taking it to be 1 outside $e'C([0,1],B)e'$, and make fairly arbitrary definitions of partial isometries, without destroying the estimates we obtained in the previous step. We need this flexibility to match things up properly at the endpoints of our homotopy.

Recall our initial assumption that $\mathrm{ev}_\alpha \circ \varphi_t$ is a constant function of α for $\alpha \in [0,1/3]$ and also for $\alpha \in [2/3,1]$. All our estimates therefore still hold if we redefine every element of $C([0,1],B)$ that appeared above to take same value on $[0,1/3]$ that it already does at $1/3$, and to take the same value on $[2/3,1]$ that it already does at $2/3$. We thus assume that everything is constant on $[0,1/3]$ and also on $[2/3,1]$.

The projections $e'(0)$ and $p(0)$ are both elements of the hereditary subalgebra of B generated by $\mathrm{ev}_0 \circ \varphi_t((u-1)\otimes 1)$, since $\mathrm{ev}_0 \circ \varphi_t$ is a homomorphism. Since $\mathrm{ev}_0 \circ \varphi_t = \psi_0|_{C_0(S^1-\{1\})\otimes \mathcal{O}_m}$, it follows that

$$\psi_0(1) > e'(0) \quad \text{and} \quad \psi_0(1) > p(0).$$

Similarly

$$\psi_1(1) > e'(1) \quad \text{and} \quad \psi_1(1) > p(1).$$

Note that

$$[\psi_0(1)] = [\psi_1(1)] \text{ in } K_0(B),$$

by (H5), so

$$[1-\psi_0(1)] = [1-\psi_1(1)] \quad \text{and} \quad [\psi_0(1) - e'(0)] = [\psi_1(1) - e'(1)].$$

Furthermore, $1-\psi_0(1)$, $1-\psi_1(1)$, $\psi_0(1)-e'(0)$ and $\psi_1(1)-e'(1)$ are all nonzero. Also $e'(\alpha)-e(\alpha)$ is always nonzero. Standard methods thus yield a unitary path $\alpha \to x(\alpha)$ such that $x(\alpha)$ is constant on $[0,1/3]$ and on $[2/3,1]$, $x(\alpha)e'(\alpha)x(\alpha)^*$ and $x(\alpha)e(\alpha)x(\alpha)^*$ are constant, and

$$x(0)\psi_0(1)x(0)^* = x(1)\psi_1(1)x(1)^*.$$

Conjugating everything by this path (and then forgetting it), we may assume, in addition to everything else, that $\psi_0(1) = \psi_1(1)$ and that e' and e are constant. Let q_0 be the constant projection with value $\psi_0(1) = \psi_1(1)$.

To construct ρ, we begin by choosing a path $\alpha \to y(\alpha)$ in $U_0(q_0(0)Bq_0(0))$, defined for $\alpha \in [0,1/3]$, such that

$$e(\alpha)y(\alpha) = y(\alpha)e(\alpha) = e(\alpha) \quad \text{and} \quad y(0)e'(0) = d(0).$$

(This is possible because the partial isometry $d(0)-e(0)$ from $e'(0)-e(0)$ to $p(0)-e(0)$ can be extended to a unitary in $U_0\left([q_0(0)-e(0)]B[q_0(0)-e(0)]\right)$.) Next, observe that, at $\alpha = 0$ (which we suppress in the notation),

$$\begin{aligned} \|e'_j\psi_0(1\otimes s_j)ye' - z_j\| &= \|(e'_j\tilde{w}'_{j1})\left(\tilde{w}'_{1j}\psi_0(1\otimes s_j)d\right) - e'_j\overline{w}_1 d\| \\ &= \|e'_{j1}\varphi_t(g'(u-1)\otimes s_1)d - e'_j\overline{w}_1 d\| \\ &\leq \|e'_{j1}\|\,[\|\varphi_t(g'(u-1)\otimes s_1) - w'_1\| + \|e'_1 w'_1 - \overline{w}_1\|]\,\|d\| \\ &< \delta + \beta_4(\delta''). \end{aligned}$$

This uses, among other things, $e_j'\tilde{w}_{j1}' = e_{j1}'e_1' = e_{j1}'$ and (17). Therefore Lemma 2.1 (5) yields a homomorphism $\tau_0 : \mathcal{O}_m \to B$ such that $\tau_0(1) = q_0(0) - e'(0)$ and

$$\|\tau_0(s_j) + z_j(0) - \psi_0(1 \otimes s_j)y(0)\| < \beta_5(\delta + \beta_4(\delta'')) < \min(\varepsilon, \eta). \tag{32}$$

(See (6).) For $\alpha \in [0, 1/3]$, we now define $\tau_\alpha = \tau_0$ and

$$\rho(s_j)(\alpha) = (\tau_0(s_j) + z_j(0))y(\alpha)^* = (\tau_\alpha(s_j) + z_j(\alpha))y(\alpha)^*.$$

A similar construction at $\alpha = 1$ yields $y(\alpha)$ for $\alpha \in [2/3, 1]$ and a homomorphism $\tau_1 : \mathcal{O}_m \to B$ such that $\tau_1(1) = q_0(1) - e'(1)$ and

$$\|\tau_1(s_j) + z_j(1) - \psi_1(1 \otimes s_j)y(1)\| < \min(\varepsilon, \eta). \tag{33}$$

For $\alpha \in [2/3, 1]$ we define $\tau_\alpha = \tau_1$ and

$$\rho(s_j)(\alpha) = (\tau_\alpha(s_j) + z_j(\alpha))y(\alpha)^*.$$

In $KK^0(\mathcal{O}_m, B)$, we now have

$$[\mathrm{ev}_0 \circ \rho_0] = [\mathrm{ev}_1 \circ \rho_0].$$

Furthermore, the estimates (32) and (33), and the choice of η at the beginning of the proof, imply that for $i = 0, 1$ we have $[\mathrm{ev}_i \circ \rho_i] = [\psi_i|_{\mathbf{C}\otimes\mathcal{O}_m}]$ in $KK^0(\mathcal{O}_m, B)$. Therefore

$$\begin{aligned} [\tau_{1/3}] + [\mathrm{ev}_{1/3} \circ \rho_0] &= [\mathrm{ev}_{1/3} \circ \rho] = [\mathrm{ev}_0 \circ \rho] = [\psi_0|_{\mathbf{C}\otimes\mathcal{O}_m}] \\ &= [\psi_1|_{\mathbf{C}\otimes\mathcal{O}_m}] = [\mathrm{ev}_1 \circ \rho] = [\tau_{2/3}] + [\mathrm{ev}_{2/3} \circ \rho_0]. \end{aligned}$$

Thus $[\tau_{1/3}] = [\tau_{2/3}]$. Since m is even, $(1 - e'(0))B(1 - e'(0))$ is purely infinite, and $\tau_{1/3}(1), \tau_{2/3}(1) < 1 - e'(0)$, Lemma 2.9 implies that $\tau_{1/3}$ is homotopic to $\tau_{2/3}$. We let $\alpha \to \tau_\alpha$ be a homotopy, defined for $\alpha \in [1/3, 3/2]$, with $\tau_{1/3}$ and $\tau_{2/3}$ as already defined. We define

$$\rho(s_j)(\alpha) = \tau_\alpha(s_j) + z_j(\alpha) \tag{34}$$

for $\alpha \in [1/3, 2/3]$.

It is now easy to define the unitary v. Let $q(\alpha) = \rho(1)(\alpha)$, which is equal to $q_0(\alpha)$ for $\alpha \notin [1/3, 2/3]$, and in any case is equal to $\tau_\alpha(1) + e_0'(\alpha)$. Set

$$v = v_0 + q - e'. \tag{35}$$

Step 9: We now verify the conclusion of the lemma for these choices of v and ρ.

We start with (C1). On $[1/3, 2/3]$, we know that $\tau(s_j)$ commutes with $q - e'$, so (30), (34), and (35) imply

$$\|v\rho(s_j) - \rho(s_j)v\| = \|v_0 z_j - z_j v_0\| < \varepsilon.$$

On $[0, 1/3]$, we first observe that the relations

$$ye = ey = e \quad \text{and} \quad v(q - e) = (q - e)v = q - e$$

(see (28)) imply that y^* commutes with v. Therefore

$$\begin{aligned} v\rho(s_j) - \rho(s_j)v &= v(\tau(s_j) + z_j)y^* - (\tau(s_j) + z_j)y^* v \\ &= [v(\tau(s_j) + z_j) - (\tau(s_j) + z_j)v]y^*. \end{aligned}$$

The term in brackets has norm at most ε for the same reason as above, and $\|y^*\| = 1$, so the estimate is verified here too. The same argument shows that it holds on $[2/3, 1]$ as well.

Next we do (C2). We have $v - \rho(1) = v_0 - e'$, so

$$\|(v - \rho(1)) - \varphi_t((u-1) \otimes 1)\| = \|v_0 - e' - \varphi_t((u-1) \otimes 1)\| < \varepsilon,$$

by Step 7. On the interval $[1/3, 2/3]$, we have $(v - \rho(1))\rho(s_j) = (v_0 - e')\rho_0(s_j)$, so the inequality $\|(v - \rho(1))\rho(s_j) - \varphi_t((u-1) \otimes s_j)\| < \varepsilon$ follows immediately from Step 7. On $[0, 1/3]$, we have

$$\begin{aligned} &(v - \rho(1))\rho(s_j) - \varphi_t((u-1) \otimes s_j) \\ &\quad = (v_0 - e')(\tau(s_j) + z_j)y^* - \varphi_t((u-1) \otimes s_j) \\ &\quad = (v_0 - e')z_j y^* - \varphi_t((u-1) \otimes s_j). \end{aligned}$$

Now $z_j y^*$ differs from z_j only that, in the definition, d has been replaced by $\tilde{d} = dy^*$. But dy^* satisfies the properties required for $\tilde{d}$ in Steps 6 and 7, so Step 7 again gives

$$\|(v - \rho(1))\rho(s_j) - \varphi_t((u-1) \otimes s_j)\| < \varepsilon.$$

The same argument applies to $[2/3, 1]$.

Finally, we do (C3). Using (32), we have

$$\|\rho_i(s_j) - \psi_i(1 \otimes s_j)\| = \|(\tau_i(s_j) + z_j(i))y(i)^* - \psi_i(1 \otimes s_j)\| < \varepsilon.$$

Also,

$$\begin{aligned} \|\psi_i(u \otimes 1) - v(i)\| &= \|\varphi_t((u-1) \otimes 1)(i) - (v(i) - q(i))\| \\ &\leq \|\varphi_t((u-1) \otimes 1) - (v - \rho(1))\| < \varepsilon, \end{aligned}$$

as has already been shown in the proof of (C2). ∎

The purpose of the following lemma is to show that we can always assume that hypotheses (H2) and (H3) of Lemma 2.10 hold.

2.11 Lemma Let $t \mapsto \varphi_t$ be an asymptotic morphism, assumed linear, contractive, and *-preserving, from $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ to a C^*-algebra A. Let $M_m \subset \mathcal{O}_m$ be the subalgebra generated by the elements $s_i s_j^*$. Then there exists a constant M and an asymptotic morphism $t \mapsto \psi_t$ from $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ to A such that:

(1) Each ψ_t is linear, *-preserving, and satisfies $\|\psi_t\| \leq M$. (We demand neither contractivity nor positivity.)

(2) $\psi_t|_{C_0(S^1 \setminus \{1\}) \otimes M_m}$ is a homomorphism for each t.

(3) Whenever $a \in C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ is actually in $C_0(S^1 - \{1\}, p_i \mathcal{O}_m p_j)$, then $\psi_t(a^*)\psi_t(a)$ is in the hereditary subalgebra of A generated by $\psi_t(C_0(S^1 \setminus \{1\}, \mathbf{C} p_j))$.

(4) $\lim_{t \to \infty}(\psi_t(a) - \varphi_t(a)) = 0$ for all $a \in C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$.

Moreover, if A has the form $C([0,1], B)$ for some C^*-algebra B, and if $\mathrm{ev}_i \circ \varphi_t$ is a homomorphism for $i = 0, 1$ and all t, then $t \mapsto \psi_t$ can be chosen to satisfy $\mathrm{ev}_i \circ \psi_t = \mathrm{ev}_i \circ \varphi_t$ for $i = 0, 1$ and all t.

Proof: We will only do the case involving homotopies with homomorphisms at the ends of the homotopy. For simplicity, we will write $\varphi_{\alpha,t}$ for $\mathrm{ev}_\alpha \circ \varphi_t$, and similarly for ψ (as we construct it). By a reparametrization of the homotopy, depending on t, we can assume without changing the limiting behavior at ∞ that

$$\varphi_{\alpha,t} = \varphi_{0,t} \quad \text{for } 0 \leq \alpha \leq \min(1/t, 1/3)$$

and

$$\varphi_{\alpha,t} = \varphi_{1,t} \quad \text{for } \max(1 - 1/t, 2/3) \leq \alpha \leq 1.$$

From here, we will only construct $\psi_{\alpha,t}$ for t greater than some T, omitting the argument needed to extend the construction back to smaller values of t while still preserving the condition $\psi_{i,t} = \varphi_{i,t}$ for $i = 0, 1$.

Recall (Definition 3.4 of [Lr1]) that a system (G, R) of C^*-algebra generators and relations is *exactly stable* if for each $\varepsilon > 0$ there is $\delta > 0$ such that there is a homomorphism $\sigma_\delta : C^*(G, R) \to C^*(G, R_\delta)$ (notation explained below) with

$$\|\sigma_\delta(g) - g^{(\delta)}\| < \varepsilon \quad \text{and} \quad \pi_\delta(\sigma_\delta(g)) = g$$

for each $g \in G$. Here $C^*(G, R)$ is the (not necessarily unital) universal C^*-algebra on the generators G satisfying the relations R. The relations R_δ are the same as the relations R, except "softened" by δ. If, following Loring's conventions, $G = \{g_1, \ldots, g_\nu\}$ and R consists of the relations $\|g_i\| \leq 1$ for $i = 1, \ldots, \nu$ and relations of the form $p_j(g_1, \ldots, g_\nu) = 0$ for polynomials p_j $(j = 1, \ldots, \mu)$ in ν noncommuting variables and their noncommuting adjoints (making 2ν noncommuting variables in all), then R_δ consists of the relations $\|g_i\| \leq 1 + \delta$ and $\|p_j(g_1, \ldots, g_\nu)\| \leq \delta$. Unless otherwise specified, we will in fact follow this convention. To avoid confusion, if $g \in G$ we denote the corresponding generator of $C^*(G, R_\delta)$ by $g^{(\delta)}$. We furthermore let $\pi_\delta : C^*(G, R_\delta) \to C^*(G, R)$ be the canonical map sending $g^{(\delta)}$ to g.

One easily checks that $C_0(S^1 \setminus \{1\})$ is generated by exactly stable relations. It therefore follows from Theorem 5.7 of [Lr1] that $C_0(S^1 \setminus \{1\}) \otimes M_m$ is generated by exactly stable relations. Let (G, R) be an exactly stable system of generators and relations for this algebra, of the form described above. For any C^*-algebra A and $a_1, \ldots, a_\nu \in A$, we define $\delta(a_1, \ldots, a_\nu)$ to be the smallest number δ for which the a_i satisfy the relations R_δ. That is, $\delta(a_1, \ldots, a_\nu)$ is the smallest number δ such that

$$\|a_i\| \leq 1 + \delta \quad \text{and} \quad \|p_j(a_1, \ldots, a_\nu)\| \leq \delta$$

for all i and j. Now choose a strictly decreasing sequence $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that $\varepsilon_n \to 0$ and:

(1) There is a homomorphism $\sigma_{\varepsilon_{n+1}} : C^*(G, R) \to C^*(G, R_{\varepsilon_{n+1}})$ with

$$\|\sigma_{\varepsilon_{n+1}}(g) - g^{(\varepsilon_{n+1})}\| < \varepsilon_n \quad \text{and} \quad \pi_{\varepsilon_{n+1}}(\sigma_{\varepsilon_{n+1}}(g)) = g$$

for $g \in G$.

(2) If $a_1, \ldots, a_\nu \in A$ satisfy the relations R, and $\|a_i - b_i\| < \varepsilon_{n+1}$ for all i, then $\delta(b_1, \ldots, b_\nu) \leq \varepsilon_n$.

For simplicity we write σ_n for σ_{ε_n}, and we define π_n, $g^{(n)}$, and R_n following the same convention. We further let $\pi_{m,n} : C^*(G, R_m) \to C^*(G, R_n)$ be the obvious projection map. We let $\delta_{\alpha,t} = \delta(\varphi_{\alpha,t}(g_1), \ldots, \varphi_{\alpha,t}(g_\nu))$, and, for $\delta_{\alpha,t} \leq \varepsilon_n$, we let $\kappa_n^{\alpha,t} : C^*(G, R_n) \to B$ be the map sending $g^{(n)}$ to $\varphi_{\alpha,t}(g)$. Note that $\delta_{\alpha,t}$ is a continuous function of α and t.

Since $t \mapsto \varphi_t$ is an asymptotic morphism, it follows that $\delta_{\alpha,t} \to 0$ uniformly in α as $t \to \infty$. Choose T so large that $\delta_{\alpha,t} \leq \varepsilon_4$ whenever $t \geq T$.

Let α and $t \geq T$ satisfy $\varepsilon_{2n} \geq \delta_{\alpha,t} \geq \varepsilon_{2n+2}$ for some n. Necessarily $n \geq 2$. Let $s \in [0, 1]$ satisfy $\delta_{\alpha,t} = s\varepsilon_{2n} + (1 - s)\varepsilon_{2n+2}$. Observe that

$$\|\sigma_{2n}(g_i) - (sg_i^{(2n)} + (1 - s)\sigma_{2n}(g_i))\| \leq \varepsilon_{2n-1}.$$

Thus the elements $sg_i^{(2n)} + (1 - s)\sigma_{2n}(g_i)$ satisfy the relations R_{2n-2}, and so there is a homomorphism $\tau_{2n,s} : C^*(G, R_{2n-2}) \to C^*(G, R_{2n})$ sending the generators $g_i^{(2n-2)}$ to these elements. Now define $\psi_{\alpha,t}^{(0)} : C_0(S^1 \setminus \{1\}) \otimes M_m \to B$ to be the composite

$$C_0(S^1 \setminus \{1\}) \otimes M_m \xrightarrow{\cong} C^*(G, R) \xrightarrow{\sigma_{2n-2}} C^*(G, R_{2n-2}) \xrightarrow{\tau_{2n,s}} C^*(G, R_{2n}) \xrightarrow{\kappa_{2n}^{\alpha,t}} B.$$

If instead $\delta_{\alpha,t} = 0$, then define $\psi_{\alpha,t}^{(0)} = \varphi_{\alpha,t}$.

We claim that $(\alpha, t) \mapsto \psi_{\alpha,t}^{(0)}$ is continuous (in the topology of pointwise convergence), and that for $a \in C_0(S^1 \setminus \{1\}) \otimes M_m$ we have $\psi_{\alpha,t}^{(0)}(a) - \varphi_{\alpha,t}(a) \to 0$ uniformly in α as $t \to \infty$. This claim will follow if we can

show that the two possible definitions of $\psi^{(0)}_{\alpha,t}$ agree when $\delta_{\alpha,t} = \varepsilon_{2n+2}$ for some n, and that for $g \in G$ we have $\|\psi^{(0)}_{\alpha,t}(g) - \varphi_{\alpha,t}(g)\| < 2\varepsilon_{2n-3}$ when $\varepsilon_{2n} \geq \delta_{\alpha,t} \geq \varepsilon_{2n+2}$. (The second estimate implies both continuity when $\delta_{\alpha,t} = 0$ and the desired limiting behavior as $t \to \infty$. Note that standard arguments show it suffices to check estimates of this sort on a set of generators.)

We do the first part of this claim first. From the definition based on the interval $[\varepsilon_{2n+2}, \varepsilon_{2n}]$, we obtain

$$\psi^{(0)}_{\alpha,t} = \kappa^{\alpha,t}_{2n} \circ (\sigma_{2n} \circ \pi_{2n-2}) \circ \sigma_{2n-2} = \kappa^{\alpha,t}_{2n} \circ \sigma_{2n}.$$

From the definition based on the interval $[\varepsilon_{2n+4}, \varepsilon_{2n+2}]$, we obtain

$$\psi^{(0)}_{\alpha,t} = \kappa^{\alpha,t}_{2n+2} \circ \pi_{2n,2n+2} \circ \sigma_{2n} = \kappa^{\alpha,t}_{2n} \circ \sigma_{2n}.$$

We now do the second part of the claim. Let $\varepsilon_{2n} \geq \delta_{\alpha,t} \geq \varepsilon_{2n+2}$, and let s be as before. For $i = 1, \ldots, \nu$ we have

$$\begin{aligned}
&\|\psi^{(0)}_{\alpha,t}(g_i) - \varphi_{\alpha,t}(g_i)\| \\
&\quad = \|\kappa^{\alpha,t}_{2n} \circ \tau_{2n,s} \circ \sigma_{2n-2}(g_i) - \kappa^{\alpha,t}_{2n}(g_i^{(2n)})\| \\
&\quad \leq \|\tau_{2n,s} \circ \sigma_{2n-2}(g_i) - \tau_{2n,s}(g_i^{(2n-2)})\| + \|\tau_{2n,s}(g_i^{(2n-2)}) - g_i^{(2n)}\| \\
&\quad \leq \|\tau_{2n,s}\|\varepsilon_{2n-3} + (1-s)\varepsilon_{2n-1} < 2\varepsilon_{2n-3},
\end{aligned}$$

as desired.

We now have to extend $\psi^{(0)}_{\alpha,t}$ to the whole of $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ in such a way that the conditions (1), (3), and (4) are satisfied. Let $e_{ij} = s_i s_j^*$; this defines a system of matrix units in M_m. Let ω_0 be any state on $e_{11}\mathcal{O}_m e_{11}$. Using the isomorphism $M_m \otimes e_{11}\mathcal{O}_m e_{11} \cong \mathcal{O}_m$, we obtain from ω_0 a bounded linear *-preserving map $\omega : \mathcal{O}_m \to M_m$ such that $\omega|_{M_m} = \mathrm{id}_{M_m}$. Define $\gamma_0, \gamma_{ij} : C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m \to C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ by $\gamma_0(a)(\zeta) = \omega(a(\zeta))$ and $\gamma_{ij}(a)(\zeta) = e_{ii}a(\zeta)e_{jj} - \omega(e_{ii}a(\zeta)e_{jj})$. These maps are the projections for a Banach space internal direct sum decomposition

$$C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m = C_0(S^1 \setminus \{1\}) \otimes M_m \oplus \bigoplus_{i,j} Q_{ij},$$

where Q_{ij} is the range of γ_{ij}.

For $s \in [1/\pi, \infty)$ define $f_s \in C_0(S^1 \setminus \{1\})$ by

$$f_s(\exp(i\theta)) = \begin{cases} s\theta & 0 \leq \theta \leq 1/s \\ 1 & 1/s \leq \theta \leq 2\pi - 1/s \\ s(2\pi - \theta) & 2\pi - 1/s \leq \theta \leq 2\pi. \end{cases}$$

Further let f_∞ be the constant function 1. Note that $s \mapsto f_s$, for $s \in [1/\pi, \infty)$, is a continuously indexed approximate identity for $C_0(S^1 \setminus \{1\})$.

We will choose a suitable continuous function $s : [0,1] \times [T, \infty] \to [1/\pi, \infty]$ such that $s(\alpha, t) = \infty$ when $\alpha = 0$ or 1 but not otherwise. We then define ψ as follows. For $\frac{1}{2}\min(1/t, 1/3) \leq \alpha \leq \frac{1}{2}\max(1 - 1/t, 2/3)$ we set

$$\psi_{\alpha,t}(a) = \begin{cases} \psi^{(0)}_{\alpha,t}(a) & a \in C_0(S^1 \setminus \{1\}) \otimes M_m \\ \psi^{(0)}_{\alpha,t}(f_{s(\alpha,t)} \otimes e_{ii})\varphi_{\alpha,t}(a)\psi^{(0)}_{\alpha,t}(f_{s(\alpha,t)} \otimes e_{jj}) & a \in Q_{ij}. \end{cases}$$

If instead $0 \leq \alpha \leq \frac{1}{2}\min(1/t, 1/3)$ or $\frac{1}{2}\max(1 - 1/t, 2/3) \leq \alpha \leq 1$, then we define

$$\psi_{\alpha,t}(a) = \begin{cases} \psi^{(0)}_{\alpha,t}(a) & a \in C_0(S^1 \setminus \{1\}) \otimes M_m \\ \varphi_{\alpha,t}((f_{s(\alpha,t)} \otimes e_{ii})a(f_{s(\alpha,t)} \otimes e_{jj})) & a \in Q_{ij}. \end{cases}$$

The definitions agree on the overlaps because, if $\alpha = \frac{1}{2}\min(1/t, 1/3)$ or $\alpha = \frac{1}{2}\max(1 - 1/t, 2/3)$, then $\psi^{(0)}_{\alpha,t} = \varphi_{\alpha,t}|_{C_0(S^1\setminus\{1\})\otimes M_m}$ and $\varphi_{\alpha,t}$ is a homomorphism. There is also no problem with the case $s(\alpha,t) = \infty$, because that only occurs when (α,t) is in the interior of the set for which the second definition applies. With this choice, the conditions (1)–(3) of the conclusion are satisfied, regardless of the choice of s. (The constant M depends on the norms of the maps involved in the Banach space direct sum definition used above.) It remains only to choose s properly so as to ensure that conclusion (4) is satisfied, and so that if a, $b \in C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ then $\|\psi_t(ab) - \psi_t(a)\psi_t(b)\| \to 0$ as $t \to \infty$.

Before starting this, we first observe that the convergence as $t \to \infty$,

$$\|\psi^{(0)}_{\alpha,t}(a) - \varphi_{\alpha,t}(a)\| \to 0 \quad \text{and} \quad \|\varphi_{\alpha,t}(ab) - \varphi_{\alpha,t}(a)\varphi_{\alpha,t}(b)\| \to 0$$

(for $a \in C_0(S^1 \setminus \{1\}) \otimes M_m$ in the first expression and a, $b \in C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ in the second), is uniform in $\alpha \in [0,1]$ and also in a and b as long as a and b are restricted to compact subsets of the appropriate domains. This follows from pointwise convergence together with the fact that the maps involved are linear and bounded uniformly in t and α.

Choose a countable subset $\{a_1, a_2, \ldots\}$ of $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ whose linear span is dense in $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ and such that each a_k is either in some Q_{ij} or in $C_0(S^1 \setminus \{1\}) \otimes M_m$. Choose a strictly increasing sequence $s_1 < s_2 < \cdots$ of positive real numbers such that $s_n \to \infty$ and

$$\|(f_s \otimes 1)a_k(f_s \otimes 1) - a_k\| < 1/n$$

for $s \geq s_n$ and $k = 1, \ldots, n$. Now choose a strictly increasing sequence $t_1 < t_2 < \cdots$ of positive real numbers such that $t_n \to \infty$, and such that the following estimates are satisfied. For $k = 1, \ldots, n$, if $a_k \in C_0(S^1 \setminus \{1\}) \otimes M_m$, then we require

$$\|\psi^{(0)}_{\alpha,t}(a_k) - \varphi_{\alpha,t}(a_k)\| \leq 1/n$$

for $t \geq t_n$ and all α, while if $a_k \in Q_{ij}$, then we require

$$\|\psi^{(0)}_{\alpha,t}(f_s \otimes e_{ii}) - \varphi_{\alpha,t}(f_s \otimes e_{ii})\| \leq 1/n, \quad \|\psi^{(0)}_{\alpha,t}(f_s \otimes e_{jj}) - \varphi_{\alpha,t}(f_s \otimes e_{jj})\| \leq 1/n,$$

and

$$\|\varphi_{\alpha,t}((f_s \otimes e_{ii})a_k(f_s \otimes e_{jj})) - \varphi_{\alpha,t}(f_s \otimes e_{ii})\varphi_{\alpha,t}(a_k)\varphi_{\alpha,t}(f_s \otimes e_{jj})\| \leq 1/n$$

for $t \geq t_n$, $s \in [s_{n-1}, s_{n+2}]$, and all α. Now let s be a continuous function satisfying:

(1) $s(\alpha,t) \geq s_n$ whenever $t \geq t_n$.

(2) $s(\alpha,t) = \infty$ for $\alpha = 0$ or 1 and all t.

(3) $s(\alpha,t) \leq s_{n+2}$ whenever $t \leq t_{n+1}$ and $\frac{1}{2}\min(1/t, 1/3) \leq \alpha \leq \frac{1}{2}\max(1 - 1/t, 2/3)$.

It is now immediate that if $a_k \in C_0(S^1 \setminus \{1\}) \otimes M_m$, then $\|\psi^{(0)}_{\alpha,t}(a_k) - \varphi_{\alpha,t}(a_k)\| \to 0$ as $t \to \infty$, uniformly in α. On the other hand, if $a_k \in Q_{ij}$, then $(f_s \otimes e_{ii})a_k(f_s \otimes e_{jj}) = (f_s \otimes 1)a_k(f_s \otimes 1)$. Using this and the fact that $\varphi_{\alpha,t}$ is contractive, estimates show that for $n \geq k$ we have

$$\|\psi^{(0)}_{\alpha,t}(a_k) - \varphi_{\alpha,t}(a_k)\| \leq 2(\|a_k\| + 1)/n$$

when the first definition of $\psi_{\alpha,t}$ applies, and

$$\|\psi^{(0)}_{\alpha,t}(a_k) - \varphi_{\alpha,t}(a_k)\| \leq 1/n$$

when the second definition applies. Thus, in either case,
$\|\psi^{(0)}_{\alpha,t}(a_k) - \varphi_{\alpha,t}(a_k)\| \to 0$ as $t \to \infty$, uniformly in α.

It follows from linearity that conclusion (4) is satisfied for a in a dense subset of $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$. Since $\sup_t \|\varphi_t\|$ and $\sup_t \|\psi_t\|$ are finite, a standard argument implies that it in fact holds for all $a \in C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$.

It remains only to show that $\|\psi_t(ab) - \psi_t(a)\psi_t(b)\| \to 0$ as $t \to \infty$. The following computation, together with the estimates $\|\psi_t(a)\| \leq M\|a\|$ and $\|\varphi_t(a)\| \leq \|a\|$ for all t, shows that the required condition follows from conclusion (4):

$$\begin{aligned}
&\|\psi_t(ab) - \psi_t(a)\psi_t(b)\| \\
&\qquad \leq \|\psi_t(ab) - \varphi_t(ab)\| + \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| \\
&\qquad\quad + \|\varphi_t(a) - \psi_t(a)\|\|\varphi_t(b)\| + \|\psi_t(a)\|\|\varphi_t(b) - \psi_t(b)\|.
\end{aligned}$$

∎

3 Approximate unitary equivalence of homomorphisms

The purpose of this section is to show that if $X \subset S^1$ and φ and ψ are two injective unital approximately absorbing homomorphisms from $C(X) \otimes M_k(\mathcal{O}_m)$ (with m even) to a purely infinite simple C^*-algebra B, and if φ and ψ have the same class in KK-theory, then φ is approximately unitarily equivalent to ψ. Our first lemma enables us to reduce to the case $k = 1$. This is followed by a number of technical lemmas which combine to give the proof in this case. We then define ε-approximately injective homomorphisms, and prove several results which enable us to say something useful about homomorphisms which are not injective.

In this section, we will use the definition of KK-theory in terms of asymptotic morphisms [CH], rather than in terms of Kasparov bimodules as in [Ks]. Corollary 9 of [CH] shows that both definitions give the same group when both variables are separable and the first variable is nuclear. Since $\mathcal{O}_m$ is separable and nuclear, these conditions will hold in all cases of interest to us.

Throughout this section, all C^*-algebras will be assumed separable. Usually, B will be a purely infinite simple C^*-algebra and m will be a positive even integer.

3.1 Lemma Suppose that, for a fixed even m and a fixed subset $X \subset S^1$, the algebra $C = C(X) \otimes \mathcal{O}_m$ has the property that, for any purely infinite simple C^*-algebra B, two injective unital approximately absorbing homomorphisms $\varphi, \psi : C \to B$, satisfying $[\varphi] = [\psi]$ in $KK^0(C, B)$, are necessarily approximately unitarily equivalent. Then any matrix algebra $M_k(C)$ has the same property.

Proof: Let C be as in the statement, and let $\varphi, \psi : M_k(C) \to B$ be two unital approximately absorbing homomorphisms satisfying $[\varphi] = [\psi]$ in $KK^0(C, B)$. Let $\{e_{\mu\nu}\}$ be a system of matrix units in M_k. Then in particular $[\varphi(e_{11} \otimes 1)] = [\psi(e_{11} \otimes 1)]$ in $K_0(B)$. Since B is purely infinite, there exists a unitary $z_0 \in B$ such that $z_0\varphi(e_{11} \otimes 1)z_0^* = \psi(e_{11} \otimes 1)$. Define

$$z = \sum_{\mu=1}^{k} \psi(e_{\mu 1} \otimes 1) z_0 \varphi(e_{1\mu} \otimes 1).$$

Then z is a unitary satisfying $z\varphi(e_{\mu\nu} \otimes 1)z^* = \psi(e_{\mu\nu} \otimes 1)$. Replacing φ by $z\varphi(-)z^*$, we can assume that $\varphi|_{M_k \otimes 1} = \psi|_{M_k \otimes 1}$. Set $B_0 = \varphi(e_{11} \otimes 1)B\varphi(e_{11} \otimes 1)$. Identifying B with $M_k(B_0)$ in the obvious way, we can now assume that $\varphi = \mathrm{id}_{M_k} \otimes \varphi_0$ and $\psi = \mathrm{id}_{M_k} \otimes \psi_0$ for suitable unital homomorphisms $\varphi, \psi : C \to B_0$.

We now claim that $[\varphi_0] = [\psi_0]$ in $KK^0(C, B_0)$. Let $\sigma \in KK^0(\mathbf{C}, M_k)$ be the class of the map which sends 1 to a rank one projection. Then σ has an inverse $\tau \in KK^0(M_k, \mathbf{C})$. It follows that

$$[\varphi_0] = (\sigma \times \mathrm{id}_C) \times [\mathrm{id}_{M_k} \otimes \varphi_0] \times (\tau \times \mathrm{id}_{B_0}) = (\sigma \times \mathrm{id}_C) \times [\varphi] \times (\tau \times \mathrm{id}_{B_0})$$

as elements of $KK^0(C, B_0)$. The same calculation applies to ψ_0 and ψ. Therefore $[\varphi_0] = [\psi_0]$.

Clearly φ_0 and ψ_0 are both injective, and they are approximately absorbing by Corollary 1.10. The hypothesis on C implies that φ_0 is approximately unitarily equivalent to ψ_0. It follows immediately that φ is approximately unitarily equivalent to ψ. ∎

3.2 Lemma For any $\varepsilon > 0$, there exists $\delta > 0$ such that: If B is a purely infinite simple C^*-algebra, and if $v \in B$ is unitary and $s_1, \ldots, s_m \in B$ are isometries with

$$\sum_{j=1}^{m} s_j s_j^* = 1, \quad \|vs_j - s_j v\| < \delta, \quad \text{and} \quad v \in U_0(B),$$

then there are $\lambda_1, \dots, \lambda_l \in S^1$ and mutually orthogonal projections $p_1, \dots, p_l$ in B such that

$$\|v - \sum_{i=1}^{l} \lambda_i p_i\| < \varepsilon \quad \text{and} \quad \left[\sum_{j=1}^{m} s_j p_i s_j^*\right] = [p_i] \text{ in } K_0(B).$$

Proof: Since $v \in U_0(B)$, we use [Ph2] to choose $\lambda_1, \dots, \lambda_l \in S^1$ and mutually orthogonal projections $p_1, \dots, p_l$ such that

$$\|v - \sum_{i=1}^{l} \lambda_i p_i\| < \varepsilon/4.$$

What we need to do here is to choose the p_i properly so as to satisfy the other requirement.

We first consider the following situation: v is actually in qBq for some projection q satisfying $m[q] = [q]$ in $K_0(B)$, and v is approximated as above in such a way that $p_i \in qBq$ and

$$|\lambda_i - \lambda_{i+1}| < \varepsilon/2 \text{ for } i = 1, \dots, l.$$

However, no assumption is made on $\|vs_j - s_j v\|$.

Note that the existence of the isometries s_j implies that $m[1] = [1]$. Furthermore, if $\mathrm{sp}(v) = S^1$ and the λ_i are ordered cyclically, then the estimate above must hold. Thus, this situation covers the case $v \in B$ and $\mathrm{sp}(v) = S^1$.

Since qBq is purely infinite and simple, there is a nonzero projection $p_1' \leq p_1$ such that $[p'] = [q]$ in $K_0(qBq)$. There is then a nonzero projection $p_2'' \leq p_2$ such that $[(p - p_1') + p_2''] = [q]$; set $p_2' = (p - p_1') + p_2''$. Proceeding inductively, we obtain mutually orthogonal projections p_i' for $i = 1, \dots, l-1$ such that $[p_i'] = [q]$, $p_i' \leq p_{i-1} + p_i$ for $i = 1, \dots, l-1$, and $\sum_{i=1}^{l-2} p_i \leq \sum_{i=1}^{l-1} p_i'$. (We take $p_0 = 0$.) Finally, let $p_l' = q - \sum_{i=1}^{l-1} p_i'$.

We have $m[q] = [q]$ and $m[p_i'] = [p_i']$ for $i = 1, \dots, l-1$. Therefore

$$m[p_l'] = m([q] - \sum_{i=1}^{l-1} [p_i']) = [q] - \sum_{i=1}^{l-1} [p_i'] = [p_l']$$

too. Thus

$$\left[\sum_{j=1}^{m} s_k p_i' s_k^*\right] = m[p_i'] = [p_i']$$

for all i. Since $p_i' \leq p_{i-1} + p_i$ and $|\lambda_i - \lambda_{i+1}| < \varepsilon/2$, we conclude that

$$\|v - \sum_{i=1}^{l} \lambda_i p_i'\| \leq \|v - \sum_{i=1}^{l} \lambda_i p_i\| + \|\sum_{i=1}^{l} \lambda_i p_i - \sum_{i=1}^{l} \lambda_i p_i'\| < \varepsilon.$$

This completes the proof in this situation.

Now we prove the lemma. Let $X = \mathrm{sp}(v)$. We may assume (using the remark above) that $X \neq S^1$. Write $X = \coprod_{n=1}^{N} X_n$, the disjoint union of closed subsets X_n, in such a way that:

(1) $\mathrm{dist}(X_k, X_n) \geq \varepsilon/6$ for $k \neq n$, and

(2) For fixed n, there are $\lambda_1', \dots, \lambda_l' \in X_n$ such that $|\lambda_i' - \lambda_{i+1}'| < \varepsilon/6$ for $i = 1, \dots, l$, and such that every $\lambda \in X_n$ is within $\varepsilon/6$ of some λ_i.

Let q_n be the spectral projection for v corresponding to X_n. Then $v = \sum_{n=1}^{N} v_n$ with $v_n = q_n v q_n$ (so that $\mathrm{sp}(v_n) = X_n$ in $q_n B q_n$). For δ small enough (depending only on ε and m), if $\|vs_j - s_j v\| < \delta$, then

$$\|q_n s_j - s_j q_n\| < 1/m \text{ for } j = 1, \dots, m \text{ and } n = 1, \dots, N.$$

It follows that

$$\left\| q_n - \sum_{j=1}^{m} s_j q_n s_j^* \right\| < 1.$$

Therefore

$$\left[\sum_{j=1}^{m} s_j q_n s_j^* \right] = [q_n],$$

whence $m[q_n] = [q_n]$. We now approximate each v_n to within $\varepsilon/6$ by a unitary $\sum_i \lambda_i p_i$ with finite spectrum in $q_n B q_n$. We assume the numbers λ_i are ordered cyclically. Since there can be no gaps in $\mathrm{sp}(v_n)$ of length greater than $\varepsilon/6$, it follows that $|\lambda_i - \lambda_{i+1}| < \varepsilon/2$. It now follows from the special situation considered above that there are mutually orthogonal projections $p_i' \in q_n B q_n$ such that

$$\left\| v_n - \sum_i \lambda_i p_i' \right\| < \varepsilon \quad \text{and} \quad \left[\sum_j s_j p_i' s_j^* \right] = [p_i'].$$

We get the conclusion of the lemma by summing over n. ∎

3.3 Lemma Let m be even. For any $\varepsilon > 0$ there is $\delta > 0$ such that if a unital homomorphism φ from $\mathcal{O}_m$ to a purely infinite simple C^*-algebra B and a unitary $v \in U_0(B)$ satisfy:

$$\|v\varphi(s_j) - \varphi(s_j)v\| < \delta \quad \text{for } j = 1, \ldots, m,$$

then there is a homomorphism $\psi : C(S^1) \otimes \mathcal{O}_m \to M_2(B)$ such that:

(1) $\psi(u \otimes 1)$ has finite spectrum.

(2) $\|v \oplus v^* - \psi(u \otimes 1)\| < \varepsilon$.

(3) $\|\varphi(s_j) \oplus \varphi(s_j) - \psi(1 \otimes s_j)\| < \varepsilon$ for $j = 1, \ldots, m$.

Proof: Let δ_1, $\delta_2 > 0$. (We will choose δ_1 and δ_2 later.)

By the previous lemma, there are mutually orthogonal projections $p_1, p_2, \ldots, p_l$ and a unitary $v_0 = \sum_{i=1}^{l} \zeta_i p_i \in B$ with finite spectrum such that $\|v_0 - v\| < \delta_1/(2m)$ and, for each i, the projection

$$q_i = \sum_{j=1}^{m} \varphi(s_j) p_i \varphi(s_j)^*$$

satisfies $[q_i] = [p_i]$. Since B is purely infinite, p_i is unitarily equivalent to q_i. Furthermore,

$$\sum_{i=1}^{l} q_i = \sum_{i=1}^{l} p_i = 1,$$

so in fact there is a unitary $U \in B$ such that $U^* q_i U = p_i$ for all i. Choose $z \in U(p_1 B p_1)$ such that $[z] = -[U]$ in $K_1(p_1 B p_1) \cong K_1(B)$. Replacing U by $U[z + (1 - p_1)]$, we may assume $[U] = 0$ in $K_1(B)$, and thus that $U \in U_0(B)$.

Set $t_j^{(i)} = U^* \varphi(s_j) p_i$. Then we have

$$(t_j^{(i)})^* t_j^{(i)} = p_i \quad \text{and} \quad \sum_{j=1}^{m} t_j^{(i)} (t_j^{(i)})^* = p_i.$$

For $i \neq i'$ and any j, j', we further have

$$t_j^{(i)} t_{j'}^{(i')} = t_{j'}^{(i')} t_j^{(i)} = 0.$$

Set $t_j = \sum_{i=1}^{l} t_j^{(i)}$. Then the t_j are isometries which commute with v_0 and which generate a C^*-subalgebra of B isomorphic to $\mathcal{O}_m$. Thus we can define $\psi_1 : C(S^1) \otimes \mathcal{O}_m \to B$ by $\psi_1(u) = v_0$ and $\psi_1(s_j) = t_j$, and also define $\psi_2 : C(S^1) \otimes \mathcal{O}_m \to B$ by $\psi_2(u) = v_0^*$ and $\psi_2(s_j) = t_j$.

Define $\lambda : B \to B$ by

$$\lambda(a) = \sum_{j=1}^{m} \varphi(s_j) a \varphi(s_j)^*.$$

If δ is chosen less than $\delta_1/(4m)$, then $\|v_0\varphi(s_j) - \varphi(s_j)v_0\| < \delta_1/m$ for $j = 1, \ldots, m$. It follows that $\|\lambda(v_0) - v_0\| < \delta_1$. We compute

$$U^*\lambda(v_0)U = U^* \left(\sum_{j=1}^{m} \varphi(s_j) v_0 \varphi(s_j^*) \right) U = U^* \left(\sum_{j=1}^{m} \zeta_j q_j \right) U = \sum_{j=1}^{m} \zeta_j p_j = v_0.$$

So

$$\|U^* v_0 U - v_0\| = \|U^* v_0 U - U^* \lambda(v_0) U\| < \delta_1.$$

Theorem 2.6 of [Ln4] implies that, if δ_1 is small enough, then there are two commuting unitaries $U_0, V_0 \in B$ such that

$$\|U - U_0\| < \delta_2/2 \quad \text{and} \quad \|V - v_0\| < \delta_2/2.$$

Define $\tilde{U} = U_0 \oplus U_0, \tilde{V} = V_0 \oplus V_0^* \in M_2(B)$. Let $\mu : C(S^1 \times S^1) \to M_2(B)$ be the homomorphism sending the two canonical generators of $C(S^1 \times S^1)$ to $\tilde{U}$ and $\tilde{V}$. Notice that $\tilde{U}$ commutes with the path

$$\alpha \mapsto \begin{pmatrix} V_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} V_0^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

from 1 to $\tilde{V}$. Therefore μ is homotopic to a homomorphism with a nontrivial kernel. Letting $d : K_0(C(S^1 \times S^1)) \to C(X, \mathbf{Z})$ be the dimension map, it follows that $\mu_*|_{\ker(d)} = 0$. Since both $\tilde{U}$ and $\tilde{V}$ are in $U_0(M_2(B))$, it also follows that μ_* is zero on K_1. Theorem 4.14 of [Ln2] now implies that $\tilde{U}$ and $\tilde{V}$ can be arbitrarily closely approximated by commuting unitaries $\tilde{U'}$ and $\tilde{V'}$ with finite spectrum. We require

$$\|\tilde{U'} - \tilde{U}\| < \delta_2/2 \quad \text{and} \quad \|\tilde{V'} - \tilde{V}\| < \delta_2/2.$$

We now follow the proof of Lemma 3.1 of [Ln3]. We take the sets of isometries to be $\{U^*\varphi(s_j) \oplus U^*\varphi(s_j)\}$ (for $\{s_j\}$) and $\{\varphi(s_j) \oplus \varphi(s_j)\}$ (for $\{t_j\}$), the unitary u there to be $U \oplus U$, and the unitary w there to be $v_0 \oplus v_0^*$. We have unitaries $\tilde{U'}$ and $\tilde{V'}$ with finite spectrum such that

$$\|U \oplus U - \tilde{U'}\| < \delta_2 \quad \text{and} \quad \| \, v_0 \oplus v_0^* - \tilde{V'}\| < \delta_2.$$

Taking δ_2 as the σ at the beginning of the proof of Lemma 3.1 of [Ln3], if δ_2 is small enough, there exists a unitary $W \in A$ such that

$$\|W^*(\varphi(s_j) \oplus \varphi(s_j))W - U^*\varphi(s_j) \oplus U^*\varphi(s_j)\| < \varepsilon/2$$

for $j = 1, \ldots, m$ and

$$\|W(v_0 \oplus v_0^*) - (v_0 \oplus v_0^*)W\| < \varepsilon/2.$$

Define $\psi(a) = W(\psi_1(a) \oplus \psi_2(a))W^*$. We claim this is the required homomorphism. Since v_0 has finite spectrum, so does $\psi(u) = W(v_0 \oplus v_0^*)W^*$. The estimates above show that

$$\|W(v_0 \oplus v_0^*)W^* - v \oplus v^*\| < \varepsilon/2 + \delta_1/(2m)$$

(since $\|v - v_0\| < \delta_1/(2m)$) and

$$\|\psi(s_j) - \varphi(s_j) \oplus \varphi(s_j)\| = \|W(U^*\varphi(s_j) \oplus U^*\varphi(s_j))W^* - \varphi(s_j) \oplus \varphi(s_j)\| < \varepsilon/2.$$

We may assume δ_1 has been chosen to satisfy $\delta_1/(2m) < \varepsilon/2$, and so we are done. ∎

The following lemma essentially says that asymptotic homomorphisms from $C(S^1) \otimes \mathcal{O}_m$ can be required to be homomorphisms when restricted to $C(S^1) \otimes 1$ and $1 \otimes \mathcal{O}_m$.

3.4 Lemma Let $t \mapsto \varphi_t$ be a unital asymptotic morphism from $C(S^1) \otimes \mathcal{O}_m$ to a unital C^*-algebra B. (That is, we assume $\varphi_t(1) \to 1$ as $t \to \infty$.) Then there exists a continuous unitary path $t \mapsto v_t$ in B and a continuous path $t \mapsto \psi_t$ of unital homomorphisms from $\mathcal{O}_m$ to B such that the expressions

$$\|v\psi_t(s_j) - \psi_t(s_j)v\|, \quad \|v - \varphi_t(u \otimes 1)\| \quad \text{and} \quad \|\psi_t(s_j) - \varphi_t(1 \otimes s_j)\|$$

all converge to 0 as $t \to \infty$.

Proof: It suffices to show that v_t and ψ_t can be constructed for $t \geq t_0$ for some t_0. (We then just use the values at t_0 for $t < t_0$.)

Note that $\varphi_t(u \otimes 1)\varphi_t(u^* \otimes 1) - 1 \to 0$, since $t \to \varphi_t$ is a unital asymptotic homomorphism. We choose t_0 so large that $\varphi_t(u \otimes 1)$ is invertible for $t \geq t_0$, and set

$$v_t = \varphi_t(u \otimes 1)[\varphi_t(u \otimes 1)^*\varphi_t(u \otimes 1)]^{-1/2}.$$

Certainly $\|v - \varphi_t(u \otimes 1)\| \to 0$ as $t \to \infty$.

Similarly, if $t \geq t_0$ and t_0 is sufficiently large, we can use the exact stability of the defining relations of $\mathcal{O}_m$, as in the last paragraph of the proof of Lemma 2.1, to construct $\psi_t(s_j)$ from the elements $\varphi_t(1 \otimes s_j)$. Exact stability implies in particular that $\|\psi_t(s_j) - \varphi_t(1 \otimes s_j)\| \to 0$ as $t \to \infty$. The relation $\|v\psi_t(s_j) - \psi_t(s_j)v\| \to 0$ as $t \to \infty$ now follows from the other two relations and the fact that $t \mapsto \varphi_t$ is an asymptotic homomorphism. ∎

3.5 Lemma Let m be even, and let B be a purely infinite simple C^*-algebra. Let φ_0 and φ_1 be two injective approximately absorbing nonunital homomorphisms from $C(S^1) \otimes \mathcal{O}_m$ to B such that $[\varphi_0] = [\varphi_1]$ in $KK^0(C(S^1) \otimes \mathcal{O}_m, B)$. Then for any $\varepsilon > 0$ and $\delta > 0$ there are L, partial isometries $v_0, \ldots, v_L \in B$, and homomorphisms $\psi_0, \ldots, \psi_L : \mathcal{O}_m \to B$ such that:

(1) $v_l v_l^* = v_l^* v_l = \psi_l(1)$ for all l.

(2) $\|v_l\psi_l(s_j) - \psi_l(s_j)v_l\| < \delta$ for all l and j.

(3) $v_0 = \varphi_0(u \otimes 1)$, $v_L = \varphi_1(u \otimes 1)$, $\psi_0 = \varphi_0|_{1 \otimes \mathcal{O}_m}$, and $\psi_L = \varphi_1|_{1 \otimes \mathcal{O}_m}$.

(4) $\|v_l - v_{l-1}\| < \varepsilon$ and $\|\psi_l(s_j) - \psi_{l-1}(s_j)\| < \varepsilon$ for all l and j.

Essentially, this lemma promises the existence of a discrete version of a homotopy from φ_0 to φ_1 via approximate homomorphisms from $C(S^1) \otimes \mathcal{O}_m$ to B which are homomorphisms when restricted to $C(S^1) \otimes 1$ and $1 \otimes \mathcal{O}_m$.

Proof: Let us say that homomorphisms φ_0 and φ_1 are connected by an (ε, δ)-chain if there exist $L, v_0, \ldots, v_L$, and $\psi_0, \ldots, \psi_L$, such that the conclusions (1)–(3) hold. Note that this relation is an equivalence relation, and that homotopic homomorphisms are necessarily related in this manner.

Since $[\varphi_0(1)] = [\varphi_1(1)]$ and both are not the identity of B, there is a unitary path $t \to U_t \in B$ such that $U_0 = 1$ and $U_1^*\varphi_1(1)U_1 = \varphi_0(1)$. We may replace φ_1 by the homotopic homomorphism $U_1^*\varphi_1(-)U_1$, and thus

assume that $\varphi_0(1) = \varphi_1(1)$. Let $q \in (1-\varphi_0(1))B(1-\varphi_0(1))$ be a nonzero projection such that $q \neq 1-\varphi_1(1)$. Let $h \in qBq$ be selfadjoint and satisfy $\mathrm{sp}(h) = [0,1]$. (Since B is a non-elementary simple C^*-algebra, such an element exists by page 61 of [AS].) Then the hereditary C^*-subalgebra B_1 generated by $h + \varphi_0(1)$ is nonunital but σ-unital. It follows from [Bn1] and Theorem 1.2 (i) of [Zh1] that $B_1 \cong B \otimes \mathcal{K}$.

The maps $\varphi_0|_{C_0(S^1\setminus\{1\})\otimes\mathcal{O}_m}$ and $\varphi_1|_{C_0(S^1\setminus\{1\})\otimes\mathcal{O}_m}$, regarded as homomorphisms from $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ to B_1, have the same class in KK-theory. Since B_1 is stable, it follows from Corollary 5.2 of [DL] that there is a homotopy $t \to \mu_t^{(0)}$ of asymptotic morphisms from $C_0(S^1 \setminus \{1\}) \otimes \mathcal{O}_m$ to $B \otimes K$ with $\mu_t^{(0)} = \varphi_t|_{C_0(S^1\setminus\{1\})\otimes\mathcal{O}_m}$ for $t = 0, 1$. Applying the version of Lemma 2.11 for homotopies, we may assume that $\mu_t^{(0)}$ in addition has the properties (1)–(3) of the conclusion of that lemma. (Note that we do not get to assume that $\mu_t^{(0)}$ is contractive or positive.) Take a nonzero projection $p \in B_2 = (1 - q - \varphi_1(1))B(1 - q - \varphi_1(1))$ such that $p \neq 1-q-\varphi_1(1)$. Now let $\nu : C(S^1)\otimes\mathcal{O}_m \to pB_2p$ be the (nonunital) homomorphism constructed in Lemma 1.15. Set $\mu_t = \mu_t^{(0)} + \nu|_{C_0(S^1\setminus\{1\})\otimes\mathcal{O}_m}$. Then the asymptotic morphism $t \to \mu_t$ satisfies the hypotheses of Lemma 2.10, with the ψ_i there taken to be $\varphi_i + \nu$. (Note that the full spectrum condition, hypothesis (H6) of Lemma 2.10, is satisfied because ν is injective.)

By applying Lemma 2.10, we see that $\varphi_0 + \nu$ and $\varphi_1 + \nu$ are connected by an (ε, δ)-chain. Therefore the proof will be complete if we show that, for $i = 0, 1$, the homomorphisms $\varphi_i + \nu$ and φ_i are connected by an (ε, δ)-chain.

Since φ_i is approximately absorbing, we have

$$\varphi_i \overset{\varepsilon}{\sim} \varphi_i + \nu.$$

So φ_i and $W^*(\varphi_i+\nu)(-)W$, for some unitary $W \in (p+q+\varphi_0(1))B(p+q+\varphi_0(1))$, are connected by an (ε,δ)-chain (with $L = 1$). Since B is purely infinite, there is a unitary $V \in (1-(p+q+\varphi_i(1))\tilde{B}(1-(p+q+\varphi_i(1))$ such that $[V] = -[W]$ in $K_1(B)$. Set $U = W+V$. Then $U \in U_0(B)$. Therefore $W^*(\varphi_i+\nu)(-)W = U^*(\varphi_i+\nu)(-)U$ is actually homotopic to φ_i. ∎

3.6 Lemma Let m be even, and let B be a purely infinite simple C^*-algebra. Let $X \subset S^1$ be a closed proper subset, and let φ_0 and φ_1 be two injective nonunital homomorphisms from $C(X) \otimes \mathcal{O}_m$ to B such that $[\varphi_0] = [\varphi_1]$ in $KK^0(C(X) \otimes \mathcal{O}_m, B)$. Then for any $\varepsilon > 0$, there are L, partial isometries $v_0, \ldots, v_L \in B$, and homomorphisms $\psi_0, \ldots, \psi_L : \mathcal{O}_m \to B$ such that:

(1) $v_l v_l^* = v_l^* v_l = \psi_l(1)$ for all l.

(2) $v_l \psi_l(s_j) = \psi_l(s_j) v_l$ for all l and j.

(3) $v_0 = \varphi_0(u \otimes 1)$, $v_L = \varphi_1(u \otimes 1)$, $\psi_0 = \varphi_0|_{1\otimes\mathcal{O}_m}$, and $\psi_L = \varphi_1|_{1\otimes\mathcal{O}_m}$.

(4) $\|v_l - v_{l-1}\| < \varepsilon$ and $\|\psi_l(s_j) - \psi_{l-1}(s_j)\| < \varepsilon$ for all l and j.

(5) $\mathrm{sp}(v_l) \subset X$ for all l.

Proof: Fix $\varepsilon > 0$.

We first consider the case X finite, say $X = \{\lambda_1, \ldots, \lambda_N\}$. In this case, we will actually construct a homotopy $t \mapsto \varphi_t$ from φ_0 to φ_1. The required v_l and ψ_l will then be given by

$$v_l = \varphi_{t_l}(u \otimes 1) \quad \text{and} \quad \psi_l(a) = \varphi_{t_l}(1 \otimes a)$$

for a sufficiently fine partition $0 = t_0 < t_1 < \cdots < t_L = 1$ of $[0, 1]$.

Set $e_n = \chi_{\{\lambda_n\}}$. For $i = 0, 1$ and $n = 1, \ldots, N$ define $\psi_i^{(n)} : \mathcal{O}_m \to B$ by $\psi_i^{(n)}(a) = \varphi_i(e_n \otimes a)$. Since $[\varphi_0] = [\varphi_1]$ in $KK^0(C(X) \otimes \mathcal{O}_m, B)$, we have $[\varphi_0(e_n)] = [\varphi_1(e_n)]$. Therefore there is a path of unitaries $t \mapsto w_t$ in B such that

$$w_0^* \varphi_0(e_n) w_0 = \varphi_0(e_n) \quad \text{and} \quad w_1^* \varphi_0(e_n) w_1 = \varphi_1(e_n).$$

So, without loss of generality, we may further assume that $\varphi_0(e_n) = \varphi_1(e_n)$. Now the condition that $[\varphi_0] = [\varphi_1]$ in $KK^0(C(X) \otimes \mathcal{O}_m, B)$ implies that

$$[\psi_0^{(n)}] = [\psi_1^{(n)}] \quad \text{in } KK^0(\mathcal{O}_m, B).$$

Lemma 2.9 implies that $\psi_0^{(1)}$ is homotopic to $\psi_1^{(1)}$. Call the homotopy $t \mapsto \psi_t^{(1)}$. Choose a unitary path $t \mapsto v_t^{(1)}$ such that $v_0^{(1)} = 1$ and $(v_t^{(1)})^* \psi_0^{(1)}(1) v_t^{(1)} = \psi_t^{(1)}(1)$. Then $v_1^{(1)}$ commutes with $\varphi_1(e_1)$. Now use Lemma 2.9 to produce a homotopy $t \mapsto \psi_t^{(2)\prime}$ of homomorphisms from $\mathcal{O}_m$ to $(1 - \psi_0^{(1)}(1))B(1 - \psi_0^{(1)}(1))$, such that $\psi_0^{(2)\prime} = \psi_0^{(2)}$ and $\psi_1^{(2)\prime} = v_1^{(1)} \psi_1^{(2)}(-)(v_1^{(1)})^*$. Define $\psi_t^{(2)} = (v_t^{(1)})^* \psi_t^{(2)\prime} v_t^{(1)}$. This yields a homotopy from $\psi_0^{(2)}$ to $\psi_1^{(2)}$ whose range is orthogonal to that of $\psi_t^{(1)}$ at each t. Now let $t \mapsto v_t^{(1)}$ be a unitary path which conjugates $\psi_0^{(1)}(1) + \psi_0^{(2)}(1)$ to $\psi_t^{(1)}(1) + \psi_t^{(2)}(1)$, and construct a homotopy $t \mapsto \psi_t^{(3)}$, etc. Then define the required homotopy from φ_0 to φ_1 by $\varphi_t(f \otimes a) = \sum_n f(\lambda_n) \psi_t^{(n)}(a)$.

Now consider the case in which X contains no arc of length greater than $\varepsilon/2$. We can then write $X = \coprod_{n=1}^N X_n$, where each X_n is closed and contained in an arc of length at most $\varepsilon/2$. Let $e_n = \chi_{X_n}$, and let $u_n = e_n u e_n$. Then the e_n are mutually orthogonal projections which sum to 1, and the u_n are unitaries in $e_n C(X) e_n$ which sum to u. For each n, choose some $\lambda_n \in X_n$. Then

$$\|u - \sum_{n=1}^N \lambda_n e_n\| < \varepsilon.$$

Let φ_0', $\varphi_1' : C(X) \otimes \mathcal{O}_m \to B$ be the homomorphisms defined by

$$\varphi_i'(f \otimes a) = \sum_{n=1}^N f(\lambda_k) \varphi_i(e_n \otimes a).$$

We then have

$$\|\varphi_i'(u \otimes 1) - \varphi_i(u \otimes 1)\| = \|\varphi_i(u - \sum_{n=1}^N \lambda_n e_n)\| < \varepsilon$$

and

$$\|\varphi_i'(1 \otimes s_j) - \varphi_i(1 \otimes s_j)\| = 0.$$

It therefore suffices to prove the result for φ_0' and φ_1' in place of φ_0 and φ_1. Since φ_0' and φ_1' define injective homomorphisms from $C(X') \otimes \mathcal{O}_m \to B$, with $X' = \{\lambda_1, \ldots, \lambda_N\}$, this follows from the case already done.

Now we consider the general case. The connected components of X are all either points or closed arcs. Let $I_1, I_2, \ldots, I_N$ be the connected components of X which are arcs of length at least $\varepsilon/2$. Note that no I_n is equal to S^1. For each n, let J_n be a closed arc which contains I_n, which is at most $\varepsilon/2$ longer than I_n, and whose endpoints are not in X. We further require that the J_n be disjoint. Let $h : X \to X$ be the continuous function which is the identity on

$$X \setminus \bigcup_{n=1}^N (J_n \setminus I_n)$$

and which sends each point of $X \cap (J_n \setminus I_n)$ to the nearest endpoint of I_n. Note that $\|h(\lambda) - \lambda\| < \varepsilon$ for all $\lambda \in X$. Let φ_0', $\varphi_1' : C(X) \otimes \mathcal{O}_m \to B$ be the homomorphisms defined by

$$\varphi_i'(f \otimes a) = \varphi_i((f \circ h) \otimes a).$$

We then have, as in the previous case,

$$\|\varphi_i'(u \otimes 1) - \varphi_i(u \otimes 1)\| < \varepsilon \quad \text{and} \quad \varphi_i'(1 \otimes s_j) = \varphi_i(1 \otimes s_j).$$

Therefore, as before, we need only prove the result for φ_0' and φ_1'. We may regard them as injective maps $C(X') \otimes \mathcal{O}_m \to B$, with

$$X' = \left(X \setminus \bigcup_{n=1}^{N} J_n \right) \cup \bigcup_{n=1}^{N} I_n \subset X.$$

Now let $t \mapsto h_t$ be a homotopy of continuous maps from X' to X' such that $h_0 = \mathrm{id}_{X'}$ and h_1 is constant with value $\lambda_n \in I_n$ on each arc I_n. Then φ_i' is homotopic to the homomorphism φ_i'' given by

$$\varphi_i''(f \otimes a) = \varphi_i'((f \circ h_1) \otimes a).$$

As we saw in the proof of the case of finite X, we can replace homomorphisms by homotopic ones. Therefore it suffices to obtain the result for φ_0'' and φ_1''. These homomorphisms may be regarded as injective maps $C(X'') \otimes \mathcal{O}_m \to B$, with

$$X'' = \left(X' \setminus \bigcup_{n=1}^{N} I_n \right) \cup \{\lambda_1, \dots, \lambda_N\} \subset X'.$$

The set X'' contains no arcs of length greater than $\varepsilon/2$, so we are reduced to the previous case. ∎

3.7 Proposition Let m be even, and let B be a purely infinite simple C^*-algebra. Let X be a closed subset of S^1, and let φ_0 and φ_1 be two injective approximately absorbing homomorphisms from $C(X) \otimes M_n(\mathcal{O}_m)$ to B, with the same class in $KK^0(C(X) \otimes M_n(\mathcal{O}_m), B)$ and either both unital or both nonunital. Then φ_0 and φ_1 are approximately unitarily equivalent.

Proof: By Lemma 3.1, we may assume that $n = 1$.

We first do the nonunital case. Let $\varepsilon > 0$. Choose L, $v_0, \dots, v_L \in U(B)$, and $\psi_0, \dots, \psi_L : \mathcal{O}_m \to B$ as in Lemma 3.6, using $\varepsilon/7$ for ε and with δ chosen so small that it works in Lemma 3.3 for the choice $\varepsilon/7$ for ε. Define $v \in M_{4L}(B)$ by

$$v = v_0^* \oplus v_0 \oplus v_0^* \oplus v_0 \oplus \cdots \oplus v_{L-1}^* \oplus v_{L-1} \oplus v_{L-1}^* \oplus v_{L-1}$$

(two copies of everything, except no v_L or v_L^*). Further define $\psi : \mathcal{O}_m \to M_{4L}(B)$ by

$$\psi(a) = \psi_0(a) \oplus \psi_0(a) \oplus \psi_0(a) \oplus \psi_0(a) \oplus \cdots$$
$$\cdots \oplus \psi_{L-1}(a) \oplus \psi_{L-1}(a) \oplus \psi_{L-1}(a) \oplus \psi_{L-1}(a)$$

(four copies of everything, except no $\psi_L(a)$). Apply Lemma 3.3 to the unitary

$$v_0 \oplus v_0^* \oplus v_1 \oplus v_1^* \oplus \cdots \oplus v_{L-1} \oplus v_{L-1}^* \in U_0(M_{2L}(B))$$

and the homomorphism

$$a \mapsto \psi_0(a) \oplus \psi_0(a) \oplus \psi_1(a) \oplus \psi_1(a) \oplus \cdots \oplus \psi_{L-1}(a) \oplus \psi_{L-1}(a)$$

from $\mathcal{O}_m$ to $M_{2L}(B)$, and conjugate everything by a suitable permutation matrix, to get a homomorphism $\eta_0 : C(S^1) \otimes \mathcal{O}_m \to M_{4L}(B)$ such that:

(1) $\eta_0(u \otimes 1)$ has finite spectrum.

(2) $\|v - \eta_0(u \otimes 1)\| < \varepsilon/7$.

(3) $\|\psi(s_j) - \eta_0(1 \otimes s_j)\| < \varepsilon/7$.

Write

$$\eta_0(f \otimes a) = \sum_{i=1}^{l} f(x_i^{(0)}) \eta_i(a)$$

where $\mathrm{sp}(\eta_0(u \otimes 1)) = \{x_0^{(0)}, \ldots, x_l^{(0)}\}$ and $\eta_1, \ldots, \eta_l$ are homomorphisms from $\mathcal{O}_m$ to $M_{4L}(B)$. Since $\mathrm{sp}(v) \subset X$, condition (2) above implies that for each i there is $x_i \in X$ with $|x_i - x_i^{(0)}| < \varepsilon/7$. Define

$$\eta(f \otimes a) = \sum_{i=1}^{l} f(x_i)\eta_i(a).$$

Then (1) – (3) above still hold, but with $\varepsilon/7$ replaced by $2\varepsilon/7$.

Choose homomorphisms $\mu_i : \mathcal{O}_m \to M_{4L}(B)$ with mutually orthogonal ranges such that $[\mu_i] = -[\eta_i]$ in $KK^0(\mathcal{O}_m, B)$. (For example, let $\sigma : \mathcal{O}_m \to \mathcal{O}_m$ be a homomorphism whose class in $KK^0(\mathcal{O}_m, \mathcal{O}_m)$ is $-[\mathrm{id}]$; such a homomorphism can be constructed by using Theorem 3.1 of [Rr2] with $D = \mathcal{O}_m \otimes K$. Then set $\mu_i = \eta_i \circ \sigma$.) Then define $\mu : C(X) \otimes \mathcal{O}_m$ by

$$\mu(f \otimes a) = \sum_{i=1}^{l} f(x_i)\mu_i(a).$$

Since φ_0 and φ_1 are approximately absorbing, we have

$$\varphi_0 \overset{\varepsilon/7}{\sim} \varphi_0 \widetilde{\oplus} (\eta \oplus \mu) \quad \text{and} \quad \varphi_1 \overset{\varepsilon/7}{\sim} \varphi_1 \widetilde{\oplus} (\eta \oplus \mu).$$

To complete the proof, we therefore show that

$$\varphi_0 \oplus \eta \overset{5\varepsilon/7}{\sim} \varphi_1 \oplus \eta.$$

Let U be the permutation matrix

$$U = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Note that $UvU^* = v^*$ and $U\psi(a)U^* = \psi(a)$. Define $\tilde{\eta}(b) = U\eta(b)U^*$ for $b \in C(X) \otimes \mathcal{O}_m$. Then $\varphi_1 \oplus \eta$ is unitarily equivalent to $\tilde{\eta} \oplus \varphi_1$, so we have to show $\varphi_0 \oplus \eta \overset{5\varepsilon/7}{\sim} \tilde{\eta} \oplus \varphi_1$. Now

$$\begin{aligned} &\|(\varphi_0 \oplus \eta)(u \otimes 1) - (\tilde{\eta} \oplus \varphi_1)(u \otimes 1)\| \\ &\quad \leq \|(\varphi_0 \oplus \eta)(u \otimes 1) - v_0 \oplus v\| + \|v_0 \oplus v - v^* \oplus v_L\| \\ &\qquad + \|v^* \oplus v_L - (\tilde{\eta} \oplus \varphi_1)(u \otimes 1)\| \\ &\quad < 2\varepsilon/7 + \varepsilon/7 + 2\varepsilon/7 = 5\varepsilon/7. \end{aligned}$$

Similarly,

$$\begin{aligned} &\|(\varphi_0 \oplus \eta)(1 \otimes s_j) - (\tilde{\eta} \oplus \varphi_1)(1 \otimes s_j)\| \\ &\quad \leq \|(\varphi_0 \oplus \eta)(1 \otimes s_j) - \psi_0(s_j) \oplus \psi(s_j)\| + \|\psi_0(s_j) \oplus \psi(s_j) - \psi(s_j) \oplus \psi_L(s_j)\| \\ &\qquad + \|\psi(s_j) \oplus \psi_L(s_j) - (\tilde{\eta} \oplus \varphi_1)(1 \otimes s_j)\| \\ &\quad < 2\varepsilon/7 + \varepsilon/7 + 2\varepsilon/7 = 5\varepsilon/7. \end{aligned}$$

This shows that we do indeed have $\varphi_0 \oplus \eta \overset{5\varepsilon/7}{\sim} \varphi_1 \oplus \eta$, and completes the proof that φ_0 is approximately unitarily equivalent to φ_1.

Now we do the unital case. Let $\iota : B \to M_2(B)$ be the standard embedding in the upper left corner. Then for any $\delta > 0$, we have $\iota \circ \varphi_0 \overset{\delta}{\sim} \iota \circ \varphi_1$. Let U be an implementing unitary. Then

$$\left\| U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^* - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\|$$

is small. Therefore there exists a unitary V such that $\|U-V\|$ is small and

$$V\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}V^*=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

If δ is chosen small enough, then V will implement an approximate unitary equivalence $\iota\circ\varphi_0 \overset{\varepsilon}{\sim} \iota\circ\varphi_1$. Now V must have the form $\mathrm{diag}(V_1,V_2)$ for $V_1,V_2\in U(B)$, and it follows that V_1 implements an approximate unitary equivalence $\varphi_0 \overset{\varepsilon}{\sim} \varphi_1$. ∎

In order to handle homomorphisms which are not necessarily injective, we introduce the following definition.

3.8 Definition Let X be a compact metric space, let D be a simple C^*-algebra, and let $\varphi : C(X)\otimes D\to C$ be a homomorphism to another C^*-algebra C. Then $\ker(\varphi)=C_0(X\setminus Y)\otimes D$ for some closed subset $Y\subset X$. Define

$$\delta(\varphi)=\sup\{\mathrm{dist}(Y,x): x\in X\}.$$

Let F be a finite subset of A. We regard elements of $C(X)\otimes D$ as continuous functions from X to D. Then we say that φ is ε-*approximately injective with respect to* F if, for every $f\in F$,

$$\|f(x_1)-f(x_2)\|<\varepsilon$$

whenever $\mathrm{dist}(x_1,x_2)\le\delta(\varphi)$.

Let $A=\bigoplus_{i=1}^k C(X_i)\otimes D_i$, where each D_i is simple. Let $\varphi : A\to C$ be a homomorphism and let F be a finite subset of A. Let $\pi_i : A\to C(X_i)\otimes D_i$ be the projection on the i-th summand. We say φ is ε-approximately injective with respect to F if each $\varphi|_{C(X_i)\otimes D_i}$ is ε-approximately injective with respect to $\pi_i(F)$.

3.9 Lemma Let C be an even Cuntz-circle algebra, let G be a finite subset of C, and let B be a purely infinite simple C^*-algebra. Let $0<\varepsilon_0<\varepsilon$, and let $\varphi : C\to B$ be approximately absorbing and ε_0-approximately injective with respect to G. Then there is an injective homomorphism $\varphi_0 : C\to B$ with $[\varphi_0]=0$ in $KK^0(C,B)$ such that $\varphi \overset{\varepsilon}{\sim} \varphi\widetilde{\oplus}\varphi_0$ with respect to G.

Proof: By considering each summand separately, we may assume, without loss of generality, that $C=C(X)\otimes M_n(\mathcal{O}_m)$. Let $e\in B$ be a nonzero projection with $[e]=0$ in $K_0(B)$. Let $\varphi_0 : C\to eBe$ be a unital injective homomorphism having the same properties as φ in Lemma 1.15. Let $\psi : C\to eBe$ be a unital homomorphism as in the conclusion of Lemma 1.15, using $(\varepsilon-\varepsilon_0)/2$ in place of ε. In particular, ψ has the form

$$\psi(f\otimes a)=\sum_{i=1}^{l} f(x_i)\psi_i(a)$$

for suitable $x_i\in X$ and $\psi_i : M_n(\mathcal{O}_m)\to B$, and

$$\|\varphi_0(g)-\psi(g)\|<(\varepsilon-\varepsilon_0)/2$$

for all $g\in G$. Let $\ker(\varphi)=C_0(X\backslash Y)\otimes M_n(\mathcal{O}_m)$. Since φ is ε_0-approximately injective with respect to G, there are $y_i\in Y\subset X$ such that $\|g(x_i)-g(y_i)\|<\varepsilon_0$ for $i=1,2,...,l$ and $g\in G$. Define $\psi' : C(X)\otimes M_n(\mathcal{O}_m)\to B$ by

$$\psi'(f\otimes a)=\sum_{i=1}^{l} f(y_i)\psi_i(a)$$

for $f\in C(X)$ and $a\in M_n(\mathcal{O}_m)$. We can rewrite the formula for ψ as $\psi(g)=\sum_{i=1}^{l}\psi_i(g(x_i))$ for $g\in C(X)\otimes M_n(\mathcal{O}_m)$, and similarly for ψ' (using y_i). Therefore

$$\|\psi(g)-\psi'(g)\|=\left\|\sum_{i=1}^{l}[\psi_i(g(x_i))-\psi_i(g(y_i))]\right\|<\varepsilon_0.$$

(The terms in the sum are in orthogonal corners of eBe.) Since φ is approximately absorbing, $\varphi \overset{(\varepsilon-\varepsilon_0)/2}{\sim} \varphi \widetilde{\oplus} \psi'$ with respect to G. Therefore $\varphi \overset{\varepsilon}{\sim} \varphi \widetilde{\oplus} \varphi_0$ with respect to G. ∎

3.10 Remark It is important to note that the homomorphism $\varphi \widetilde{\oplus} \varphi_0$ in the previous lemma is approximately absorbing and injective.

The following result is the analog of Proposition 3.7 for approximately injective homomorphisms.

3.11 Proposition Let φ_0 and φ_1 be two approximately absorbing homomorphisms from an even Cuntz-circle algebra C to a purely infinite simple C^*-algebra B, with the same class in $KK^0(C, B)$ and either both unital or both nonunital. Let $F \subset C$ be a finite generating set, let $\varepsilon > 0$, and let $0 < \varepsilon_0 < \varepsilon/2$. Suppose that both φ_0 and φ_1 are ε_0-approximately injective with respect to F. Then $\varphi_0 \overset{\varepsilon}{\sim} \varphi_1$ with respect to F.

Proof: Let $\varepsilon_0 < \eta < \varepsilon/2$. By Lemma 3.9, there are injective approximately absorbing homomorphisms $\psi_1, \psi_2 : C \to B$ such that $[\psi_i] = [\varphi_i]$ in $KK^0(C, B)$, such that $\varphi_i \overset{\eta}{\sim} \psi_i$ with respect to F, and such that either both are unital or both are nonunital. Then Proposition 3.7 implies that $\psi_0 \overset{\varepsilon-2\eta}{\sim} \psi_1$. Therefore $\varphi_0 \overset{\varepsilon}{\sim} \varphi_1$. ∎

4 The existence theorem

The purpose of this section is to prove the following existence theorem, to be used in the last section in the construction of approximate intertwinings of direct systems. We will also prove several related lemmas that will be needed in the last section. We will denote the Kasparov product of α and β by $\alpha \times \beta$ whenever it is defined. In particular, we use this notation for $\alpha \in KK^*(A, B)$ and $\beta \in KK^*(B, C)$ (yielding $\alpha \times \beta \in KK^*(A, C)$), and for $\alpha \in KK^*(A_1, B_1)$ and $\beta \in KK^*(A_2, B_2)$ (yielding $\alpha \times \beta \in KK^*(A_1 \otimes A_2, B_1 \otimes B_2)$).

4.1 Theorem Let A be an even Cuntz-circle algebra, and let $B = \varinjlim B_k$ be a direct limit of even Cuntz-circle algebras, with maps $\psi_k : B_k \to B$. Let $\alpha \in KK^0(A, B)$. Then for every sufficiently large k, there exists a permanently approximately absorbing homomorphism $\varphi : A \to B_k$ such that the class $[\varphi] \in KK^0(A, B_k)$ satisfies $[\varphi] \times [\psi_k] = \alpha$. Moreover, if B is unital, and the Kasparov product $[1_A] \times \alpha$ is $[1_B]$, then φ can be chosen to be unital.

To keep down the size of some of the formulas (so that they fit on the page), we will use the following notation throughout this section.

4.2 Notation We denote by S the C^*-algebra $C(S^1)$.

Before starting any K-theory calculations, we recall from [Cu2] that $K_0(\mathcal{O}_m) \cong \mathbf{Z}/(m-1)\mathbf{Z}$ and $K_1(\mathcal{O}_m) = 0$. Since we will make extensive use of the Universal Coefficient Theorem [RS], we also note the following (well known) fact.

4.3 Lemma Let $X \subset S^1$ be compact. Then $C(X) \otimes \mathcal{O}_m$ is in the bootstrap category $\mathcal{N}$ defined in [RS] just before 1.17.

Proof: It is shown in 2.1 of [Cu1] that $\mathcal{O}_m$ is stably isomorphic to a crossed product of an AF algebra by an action of $\mathbf{Z}$. So $\mathcal{O}_m$ is in $\mathcal{N}$. Therefore so is $C(X) \otimes \mathcal{O}_m$. (See 22.3.5 (d) and (f) and Chapter 23 in [Bl2].) ∎

4.4 Lemma Let $m, n \in \mathbf{N}, m, n \geq 2$, and let $\alpha \in KK^0(\mathcal{O}_m, \mathcal{O}_n)$. Then there exists a homomorphism $\varphi : \mathcal{O}_m \to \mathcal{O}_n$ such that $[\varphi] = \alpha$.

Proof: Theorem 3.1 of [Rr2] provides $\lambda : \mathcal{O}_m \to \mathcal{K} \otimes \mathcal{O}_n$ such that $[\lambda] = \alpha$. Let $e_{11} \in \mathcal{K}$ be the projection on the first standard basis vector, and choose a partial isometry $v \in \mathcal{K} \otimes \mathcal{O}_n$ such that $vv^* = \lambda(1)$ and $v^*v \leq e_{11} \otimes 1$. Then $\varphi(a) = v^*\lambda(a)v$, regarded as a homomorphism from $\mathcal{O}_m$ to $(e_{11} \otimes 1)(\mathcal{K} \otimes \mathcal{O}_n)(e_{11} \otimes 1) \cong \mathcal{O}_n$, satisfies $[\varphi] = \alpha$. ∎

4.5 Remark It is also easy to construct φ directly, since the Universal Coefficient Theorem ([RS], Theorem 1.17) implies that the natural map $\gamma : KK^0(\mathcal{O}_m, \mathcal{O}_n) \to \mathrm{Hom}(K_0(\mathcal{O}_m), K_0(\mathcal{O}_n))$ is an isomorphism.

The next two steps are to show that elements of $KK^0(\mathcal{O}_m, S \otimes \mathcal{O}_n)$ and $KK^0(S \otimes \mathcal{O}_m, \mathcal{O}_n)$ are representable by homomorphisms (in the second case, assuming n even). In both steps, the computation of some of the KK-classes will be done by reduction to the identification of elements of $\mathrm{Ext}(\mathcal{O}_A, B)$ in [Cu4]. We therefore introduce appropriate notation.

4.6 Notation For any C^*-algebra B, let $Q(B)$ denote the stable outer multiplier algebra $M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$. In particular, let $Q = Q(\mathbf{C})$ be the Calkin algebra. If

$$0 \to I \to B \to C \to 0$$

is a short exact sequence of C^*-algebras, let $\mathrm{Ind}_i : K_i(C) \to K_{1-i}(I)$ be the connecting homomorphism in the associated six term exact sequence in K-theory. We write $\mathrm{Ind}_1(u)$ for $\mathrm{Ind}_1([u])$ for unitaries u, and similarly for classes of projections. We extend the definition of Ind_1 to partial isometries with the same initial and final projections by setting $\mathrm{Ind}_1(u) = \mathrm{Ind}_1([u + (1-p)])$ when $uu^* = u^*u = p$.

4.7 Definition (Compare Cuntz [Cu3], Section 3.) Let A be an $m \times m$ matrix with entries in $\{0, 1\}$, satisfying the condition (I) of [CK], and let $\mathcal{O}_A$ be the corresponding Cuntz-Krieger algebra. Call its canonical generating partial isometries $s_1, \ldots, s_m$, and set $p_j = s_j s_j^*$. Let B be a C^*-algebra, and let σ and τ be two extensions of $\mathcal{O}_A$ by B, regarded as homomorphisms from $\mathcal{O}_A$ to $Q(B)$. Suppose that $\sigma(p_j) = \tau(p_j)$ for all j. Define

$$d_{\sigma,\tau} \in K_0(B)^m/(1-A)K_0(B)^m$$

to be the image there of

$$(\mathrm{Ind}_1(\sigma(s_1)\tau(s_1)^*), \ldots, \mathrm{Ind}_1(\sigma(s_m)\tau(s_m)^*).$$

4.8 Theorem There is an isomorphism

$$d : \mathrm{Ext}(\mathcal{O}_A, B) \to K_0(B)^m/(1-A)K_0(B)^m,$$

determined as follows: If σ and τ are as in the previous definition, and τ lifts to a homomorphism $\tilde{\tau} : \mathcal{O}_A \to M(\mathcal{K} \otimes B)$ such that the projections $\tilde{\tau}(p_j)$ (for $j = 1, \ldots, m$) and $1 - \tilde{\tau}(1)$ are all Murray-von Neumann equivalent to 1 in $M(K \otimes B)$, then $d([\sigma]) = d_{\sigma,\tau}$.

Proof: See Section 3 of [Cu3]. ∎

4.9 Lemma The formula for $d([\sigma])$ in the previous theorem remains valid if it is merely assumed that σ and τ are related as in Definition 4.7, and that $[\tau] = 0$ in $KK^1(\mathcal{O}_A, B)$.

Proof: Let $\tau_0 : \mathcal{O}_A \to Q(B)$ be an absorbing trivial extension ([Ks], Section 7, Definition 2; note that it is also shown in the same section that τ_0 exists). Then $\sigma \oplus \tau_0$ and $\tau \oplus \tau_0$ satisfy the hypotheses of the previous theorem. It is trivial to check that

$$\mathrm{Ind}_1((\sigma \oplus \tau_0)(s_j)(\tau \oplus \tau_0)(s_j)^*) = \mathrm{Ind}_1(\sigma(s_j)\tau(s_j)^*).$$

Therefore

$$d([\sigma]) = d([\sigma \oplus \tau_0]) = d_{\sigma\oplus\tau_0,\tau\oplus\tau_0} = d_{\sigma,\tau}. \blacksquare$$

4.10 Lemma Let $m, n \in \mathbf{N}, m, n \geq 2$, and let $\alpha \in KK^0(\mathcal{O}_m, S \otimes \mathcal{O}_n)$. Then there exists a homomorphism $\varphi : \mathcal{O}_m \to S \otimes \mathcal{O}_n$ such that $[\varphi] = \alpha$.

Proof: The Universal Coefficient Theorem ([RS], Theorem 1.17) yields a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathbf{Z}}(K_0(\mathcal{O}_m), K_1(S \otimes \mathcal{O}_n)) \quad &\longrightarrow \quad KK^0(\mathcal{O}_m, S \otimes \mathcal{O}_n) \\ &\xrightarrow{\gamma} \quad \mathrm{Hom}(K_0(\mathcal{O}_m), K_0(S \otimes \mathcal{O}_n)) \to 0. \end{aligned}$$

(The two missing terms are both zero, since $K_1(\mathcal{O}_m) = 0$.)

Let H be the set of classes in $KK^0(\mathcal{O}_m, S \otimes \mathcal{O}_n)$ of homomorphisms from $\mathcal{O}_m$ to $S \otimes \mathcal{O}_n$. We claim that H is a subgroup of $KK^0(\mathcal{O}_m, S \otimes \mathcal{O}_n)$. It follows from the exact sequence above that $KK^0(\mathcal{O}_m, S \otimes \mathcal{O}_n)$ is a finite group, and therefore it suffices to prove that H is nonempty (which is trivial) and closed under addition. So let φ and ψ be homomorphisms from $\mathcal{O}_m$ to $S \otimes \mathcal{O}_n$. Let $v, w \in \mathcal{O}_n$ be isometries with orthogonal ranges. Then

$$\sigma(a) = (1 \otimes v)\varphi(a)(1 \otimes v^*) + (1 \otimes w)\psi(a)(1 \otimes w^*)$$

defines a homomorphism from $\mathcal{O}_m$ to $S \otimes \mathcal{O}_n$ whose class in $KK(\mathcal{O}_m, S \otimes \mathcal{O}_n)$ is $[\varphi] + [\psi]$.

We now show that $\gamma(H) = \mathrm{Hom}(K_0(\mathcal{O}_m), K_0(S \otimes \mathcal{O}_n))$. Let $\iota : \mathcal{O}_n \to S \otimes \mathcal{O}_n$ be the map $\iota(a) = 1 \otimes a$. Then ι_* is an isomorphism from $K_0(\mathcal{O}_n)$ to $K_0(S \otimes \mathcal{O}_n)$. Given $\alpha \in \mathrm{Hom}(K_0(\mathcal{O}_m), K_0(S \otimes \mathcal{O}_n))$, we get $\iota_*^{-1} \circ \alpha \in \mathrm{Hom}(K_0(\mathcal{O}_m), K_0(\mathcal{O}_n))$. Lemma 4.4 and Remark 4.5 provide a homomorphism $\varphi : \mathcal{O}_m \to \mathcal{O}_n$ such that $\varphi_* = \iota_*^{-1} \circ \alpha$. Then $\gamma([\iota \circ \varphi]) = \alpha$.

To complete the proof, it now suffices to show that H contains $\mathrm{Ext}^1_{\mathbf{Z}}(K_0(\mathcal{O}_m), K_1(S \otimes \mathcal{O}_n))$. Let $[\sigma_0]$ be the standard generator of $KK^1(S, \mathbf{C})$, namely the class of the extension given by the C^*-algebra of the unilateral shift. (Thus, σ_0 is to be regarded as the homomorphism from S to Q which sends the standard unitary in S to the image in Q of the unilateral shift.) Let $[\sigma]$ be its image in $KK^1(S \otimes \mathcal{O}_n, \mathcal{O}_n)$. Note that $[\sigma_0]$ has a left inverse $[\tau_0]$ in $KK^1(\mathbf{C}, S) = K_1(S)$. Therefore $[\sigma]$ has a left inverse $[\tau]$. Hence the Kasparov product with $[\sigma]$ defines surjective homomorphisms from $K_j(S \otimes \mathcal{O}_n)$ to $K_{1-j}(\mathcal{O}_n)$. Since $K_1(S \otimes \mathcal{O}_n)$ and $K_0(\mathcal{O}_n)$ are finite groups with the same cardinality, Kasparov product with $[\sigma]$ is actually an isomorphism. Since the Universal Coefficient Theorem is natural with respect to Kasparov products (see [RS]), we obtain the following commutative diagram with exact columns, in which the horizontal maps are induced by Kasparov product on the right with $[\sigma]$, and the top horizontal map is an isomorphism:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbf{Z}}(K_0(\mathcal{O}_m), K_1(S \otimes \mathcal{O}_n)) & \longrightarrow & \mathrm{Ext}^1_{\mathbf{Z}}(K_0(\mathcal{O}_m), K_0(\mathcal{O}_n)) \\
\downarrow & & \downarrow \\
KK^0(\mathcal{O}_m, S \otimes \mathcal{O}_n) & \longrightarrow & KK^1(\mathcal{O}_m, \mathcal{O}_n) \\
\downarrow \gamma & & \downarrow \gamma' \\
\mathrm{Hom}(K_0(\mathcal{O}_m), K_0(S \otimes \mathcal{O}_n)) & \longrightarrow & \mathrm{Hom}(K_0(\mathcal{O}_m), K_1(\mathcal{O}_n)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Since $K_1(\mathcal{O}_n) = 0$, the map from $\mathrm{Ext}^1_{\mathbf{Z}}(K_0(\mathcal{O}_m), K_1(S \otimes \mathcal{O}_n))$ to $KK^1(\mathcal{O}_m, \mathcal{O}_n)$ is an isomorphism. Thus, we need to find homomorphisms $\varphi : \mathcal{O}_m \to S \otimes \mathcal{O}_n$ which induce the zero map on K_0 and such that the classes $[\varphi] \times [\sigma]$ exhaust $KK^1(\mathcal{O}_m, \mathcal{O}_n)$. To do this, we identify $KK^1(\mathcal{O}_m, \mathcal{O}_n)$ with $\mathrm{Ext}(\mathcal{O}_m, \mathcal{O}_n)$, and use Theorem 4.8 and Lemma 4.9.

Let A be the $m \times m$ integer matrix with all entries equal to 1. Then $\mathcal{O}_A$ is just $\mathcal{O}_m$. Let $s_1, \ldots, s_m \in \mathcal{O}_A$ be the canonical generating isometries, and let $p_j = s_j s_j^*$, as in Definition 4.7. Choose m nonzero orthogonal projections $q_1, \ldots, q_m \in \mathcal{O}_n$ such that the K_0 class of each q_j is zero, and such that the projection $q = q_1 + \cdots + q_m$ is strictly less than 1. Then the K_0 class of q is also zero. Since $\mathcal{O}_n$ is purely infinite, there are partial isometries $t_1, \ldots, t_m \in \mathcal{O}_n$ such that $t_j t_j^* = q_j$ and $t_j^* t_j = q$. Define a homomorphism $\varphi_0 : \mathcal{O}_m \to S \otimes \mathcal{O}_n$ by $\varphi_0(s_j) = 1 \otimes t_j$.

Now let $\eta_1, \ldots, \eta_m \in K_0(\mathcal{O}_n)$. We construct a homomorphism $\varphi : \mathcal{O}_A \to S \otimes \mathcal{O}_n$ which induces the zero map on K_0 and such that

$$d([\varphi] \times [\sigma]) = (\eta_1, \ldots, \eta_m) + (1 - A)K_0(\mathcal{O}_n)^m.$$

Taking an idea from [Cu4], let u be the canonical unitary generator of S, choose projections $e_j \leq q_j$ such that $[e_j] = -\eta_j$ in $K_0(\mathcal{O}_n)$, and define $\varphi(s_j) = [u \otimes e_j + 1 \otimes (q_j - e_j)]t_j$.

The Kasparov product $[\varphi] \times [\sigma]$ is represented by the homomorphism $\sigma \circ \varphi = (\sigma_0 \otimes \mathrm{id}) \circ \varphi$, put together as follows:

$$\mathcal{O}_A \xrightarrow{\varphi} S \otimes \mathcal{O}_n \xrightarrow{\sigma_0 \otimes \mathrm{id}} Q \otimes \mathcal{O}_n \hookrightarrow Q(\mathcal{O}_n).$$

To apply Lemma 4.9, we need a comparison extension whose class is trivial. To obtain it, we merely use φ_0 in place of φ. Note that in fact $(\sigma_0 \otimes \mathrm{id}) \circ \varphi_0$ lifts. Then

$$[\varphi] \times [\sigma] = d_{(\sigma_0 \otimes \mathrm{id}) \circ \varphi, (\sigma_0 \otimes \mathrm{id}) \circ \varphi_0} = (d_1, \ldots, d_m) + (1 - A) K_0(\mathcal{O}_n)^m,$$

where

$$d_j = \mathrm{Ind}_1((\sigma \circ \varphi)(s_j)(\sigma \circ \varphi_0)(s_j^*)) = \mathrm{Ind}_1(\sigma_0(u) \otimes e_j + 1 \otimes (q_j - e_j)) = -[e_j] = \eta_j.$$

(The minus sign appears because the unilateral shift $\sigma_0(u)$ has index -1. The index is most easily computed in $Q \otimes \mathcal{O}_n$.) So

$$d([\varphi] \times [\sigma]) = (\eta_1, \ldots, \eta_m) + (1 - A) K_0(\mathcal{O}_n)^m,$$

as desired. ∎

4.11 Lemma Let $m, n \in \mathbf{N}, m, n \geq 2$, and assume n is even. Let $\alpha \in KK^0(S \otimes \mathcal{O}_m, \mathcal{O}_n)$. Then there exists a homomorphism $\varphi : S \otimes \mathcal{O}_m \to \mathcal{O}_n$ such that $[\varphi] = \alpha$.

Proof: The Universal Coefficient Theorem now yields a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathbf{Z}}(K_1(S \otimes \mathcal{O}_m), K_0(\mathcal{O}_n)) \quad &\longrightarrow \quad KK^0(S \otimes \mathcal{O}_m, \mathcal{O}_n) \\ &\xrightarrow{\gamma} \quad \mathrm{Hom}(K_0(S \otimes \mathcal{O}_m), K_0(\mathcal{O}_n)) \to 0. \end{aligned}$$

(The two missing terms are again both zero.)

As before, let H be the set of classes in $KK^0(S \otimes \mathcal{O}_m, \mathcal{O}_n)$ of homomorphisms from $S \otimes \mathcal{O}_m$ to $\mathcal{O}_n$. It follows, just as in the proof of the previous lemma, that H is a subgroup of $KK^0(S \otimes \mathcal{O}_m, \mathcal{O}_n)$. Furthermore, one checks that $\gamma(H) = \mathrm{Hom}(K_0(S \otimes \mathcal{O}_m), K_0(\mathcal{O}_n))$ in essentially the same way as there, composing maps from $\mathcal{O}_m$ to $\mathcal{O}_n$ on the right with a point evaluation map from $S \otimes \mathcal{O}_m$ to $\mathcal{O}_m$ instead of on the left with $\iota : \mathcal{O}_n \to S \otimes \mathcal{O}_n$.

It remains to show that H contains $\mathrm{Ext}^1_{\mathbf{Z}}(K_1(S \otimes \mathcal{O}_m), K_0(\mathcal{O}_n))$. We will reduce this proof to the result of the previous lemma.

We begin by constructing a homomorphism $\omega : S \otimes S \otimes \mathcal{O}_n \to \mathcal{O}_n$ which sends the Bott element to a generator of $K_0(\mathcal{O}_n)$. We take the Bott element to be the class $b = [u] \times [u] \in K_0(S \otimes S) = K_0(C(S^1 \times S^1))$, obtained as the product of two copies of the K_1-class of the standard unitary generator u of S. If B is a C^*-algebra, we then refer for convenience to $b \times [1]$ as the Bott class in $K_0(S \otimes S \otimes B)$.

The construction is essentially done in the proof of Proposition 4.2 of [Lr2]. Assume $n > 2$. (Otherwise $KK^0(S \otimes \mathcal{O}_m, \mathcal{O}_n) = 0$, and there is nothing to prove.) Let B be the unital AF algebra whose ordered K_0-group is the group G_{n-2} (defined before Corollary 3.5 in [Lr2]) and such that the class of the identity is the image of the element $(1, 1)$ in the first term of the direct limit. Let $\eta : S \otimes S \to B$ be the unitization of the map in [Lr2], gotten from Theorem 7.3 of [EL]. It sends the Bott element $b \in K_0(S \otimes S)$ to $(1, -1)$. As in [Lr2], we tensor this map with the identity on $\mathcal{O}_n$, observe that (by [Rr1]) $B \otimes \mathcal{O}_n \cong \mathcal{O}_n$, and calculate K-theory to see that $b \times [1]$ goes to a generator of $K_0(\mathcal{O}_n)$.

The automorphisms of $\mathbf{Z}/(n-1)\mathbf{Z}$ act transitively on the generators. We can therefore compose this homomorphism with a suitable homomorphism from $\mathcal{O}_n$ to $\mathcal{O}_n$, so as to obtain a homomorphism $\omega : S \otimes S \otimes \mathcal{O}_n \to \mathcal{O}_n$ which sends the Bott element to the standard generator $[1]$ of $K_0(\mathcal{O}_n)$.

Let $\sigma_0 : S \to Q$ send the standard unitary to the unilateral shift, as in the proof of Lemma 4.10. Thus $[\sigma_0]$ is a generator of $KK^1(S, \mathbf{C})$, and has a left inverse $[\tau_0] \in KK^1(\mathbf{C}, S)$, for some homomorphism $\tau_0 : \mathbf{C} \to Q(S)$. (We will make an explicit choice for τ_0 below.) Let $\sigma = \sigma_0 \otimes \mathrm{id}_{\mathcal{O}_n}$, as before. Let $\tau = \tau_0 \otimes \mathrm{id}_{\mathcal{O}_m}$. (Note that we use m instead of n.) Naturality of the Universal Coefficient Theorem now gives the following commutative

diagram with exact columns, in which the horizontal maps are Kasparov product on the left with $[\tau]$:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathbf{Z}}(K_1(S\otimes\mathcal{O}_m), K_0(\mathcal{O}_n)) & \longrightarrow & \mathrm{Ext}^1_{\mathbf{Z}}(K_0(\mathcal{O}_m), K_0(\mathcal{O}_n)) \\ \downarrow & & \downarrow \\ KK^0(S\otimes\mathcal{O}_m, \mathcal{O}_n) & \longrightarrow & KK^1(\mathcal{O}_m, \mathcal{O}_n) \\ \downarrow \gamma & & \downarrow \gamma' \\ \mathrm{Hom}(K_0(S\otimes\mathcal{O}_m), K_0(\mathcal{O}_n)) & \longrightarrow & \mathrm{Hom}(K_1(\mathcal{O}_m), K_0(\mathcal{O}_n)) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Left multiplication by τ is an isomorphism from $\ker(\gamma)$ to $KK^1(\mathcal{O}_m, \mathcal{O}_n)$, by an argument similar to the one in Lemma 4.10. To complete the proof of the lemma, it therefore suffices to let $\beta \in KK^1(\mathcal{O}_m, \mathcal{O}_n)$, and find $\psi : S\otimes\mathcal{O}_m \to \mathcal{O}_n$ which is zero on K-theory and such that $[\tau]\times[\psi] = \beta$.

As in the proof of Lemma 4.10, let $\varphi : \mathcal{O}_m \to S\otimes\mathcal{O}_n$ be a homomorphism which is zero on K-theory and such that $[\varphi]\times[\sigma] = \beta$. Further let φ_0 be the comparison homomorphism (with trivial class in KK-theory) used in that proof. Then define $\psi = \omega\circ(\mathrm{id}_S\otimes\varphi)$ and $\psi_0 = \omega\circ(\mathrm{id}_S\otimes\varphi_0)$. We claim this ψ works. To do this, we must show that

$$[\tau]\times([\mathrm{id}_S]\otimes[\varphi])\times[\omega] = \beta.$$

It is well known that there is a rank one projection $e \in M_2\otimes S\otimes S$ which represents the class $b + [1_{S\otimes S}]$. Then (depending on sign conventions) we may take τ_0 to be the extension given by the composite

$$\mathbf{C} \to M_2\otimes S\otimes S \xrightarrow{\mathrm{id}\otimes\sigma_0\otimes\mathrm{id}_S} M_2\otimes Q\otimes S \xrightarrow{\cong} Q\otimes S \hookrightarrow Q(S),$$

in which the first map sends 1 to e, and the third map is induced by an isomorphism $M_2\otimes Q \cong Q$. Then τ is the tensor product of this with $\mathrm{id}_{\mathcal{O}_m}$, which is a homomorphism from $\mathcal{O}_m$ to $Q(S\otimes\mathcal{O}_m)$.

To compute the Kasparov pairing with ψ, we compose on the left with $Q(\psi)$. This composition can be slightly rearranged to give:

$$\mathcal{O}_m \to M_2\otimes Q\otimes S\otimes\mathcal{O}_m \to M_2\otimes Q\otimes S\otimes S\otimes\mathcal{O}_n \to M_2\otimes Q\otimes\mathcal{O}_n$$
$$\hookrightarrow M_2\otimes Q(\mathcal{O}_n) \xrightarrow{\cong} Q(\mathcal{O}_n).$$

From now on, we will take τ_0 and τ to be maps to $M_2\otimes Q\otimes S$ and $M_2\otimes Q\otimes S\otimes\mathcal{O}_m$. Then the first map is τ, the second is $\mathrm{id}_{Q\otimes S}\otimes\varphi$, and the third is $\mathrm{id}_Q\otimes\omega$. As in the proof of Lemma 4.10, we use Lemma 4.9 to compute $d([Q(\psi)\circ\tau] \in K_0(\mathcal{O}_n)^m/(1-A)K_0(\mathcal{O}_n)^m$, where A is an $m\times m$ matrix of 1's. Let $(\eta_1,\dots,\eta_m)$, with $\eta_j \in K_0(\mathcal{O}_n)$, be a representative of the image of β in this group. Since $[Q(\psi_0)\circ\tau] = 0$, it suffices to compute

$$\mathrm{Ind}_1((Q(\psi)\circ\tau)(s_j)(Q(\psi_0)\circ\tau)(s_j)^*),$$

where $s_1,\dots,s_m$ are the generating isometries of $\mathcal{O}_m$. Since Ind_1 is natural, the expression above is equal to

$$\omega_*(\mathrm{Ind}_1([((\mathrm{id}\otimes\varphi)\circ\tau)(s_j)((\mathrm{id}\otimes\varphi_0)\circ\tau)(s_j)^*]))$$
$$= \omega_*(\mathrm{Ind}_1([(\tau_0\otimes\varphi)(s_j)(\tau_0\otimes\varphi_0)(s_j)^*]).$$

The elements whose indices we want are $\tau_0(e)\otimes\varphi(s_j)^*\varphi_0(s_j)$. Since Ind_i respects tensor products, the required index is

$$\mathrm{Ind}_0(\tau_0(e))\times[\varphi(s_j)\varphi_0(s_j)^* + (1-\varphi(s_js_j^*)].$$

Now recall that $b = [u]\times[u]$. Let $s \in Q$ be the unilateral shift. Then from $[e] = 1 + b$ we get

$$\mathrm{Ind}_0(\tau_0(e)) = \mathrm{Ind}_0([1]\times[1] + [s]\times[u]) = \mathrm{Ind}_0([1])\times[1] + \mathrm{Ind}_1([s])\times[u] = -[u].$$

Also, φ was constructed so as to have

$$[\varphi(s_j)\varphi_0(s_j)^* + (1 - \varphi(s_j s_j^*)] = [u] \times (-\eta_j).$$

(See the proof of Lemma 4.10.) Therefore

$$\begin{aligned}\mathrm{Ind}_1([(Q(\psi) \circ \tau)(s_j)(Q(\psi_0) \circ \tau)(s_j)^*]) \\ = \omega_*(-[u] \times [u] \times (-\eta_j)) = \omega_*(b \times \eta_j) = \eta_j.\end{aligned}$$

Since $d(\beta)$ is the image in $K_0(\mathcal{O}_n)^m/(1 - A)K_0(\mathcal{O}_n)^m$ of $(\eta_1, \ldots, \eta_m)$, we have shown that $d([Q(\psi) \circ \tau])$ is equal to the image of β, as desired. ∎

4.12 Theorem Let $m, n \in \mathbf{N}, m, n \geq 2$, and assume that n is even. Let $\alpha \in KK^0(S \otimes \mathcal{O}_m, S \otimes \mathcal{O}_n)$. Then there exists a homomorphism $\varphi : S \otimes \mathcal{O}_m \to S \otimes \mathcal{O}_n$ such that the class $[\varphi] \in KK^0(S \otimes \mathcal{O}_m, S \otimes \mathcal{O}_n)$ is equal to α.

Proof: This time, the Universal Coefficient Theorem yields the following short exact sequence, which we write vertically so that it will fit on the page:

$$\begin{array}{c} 0 \\ \downarrow \\ \mathrm{Ext}^1_{\mathbf{Z}}(K_0(S \otimes \mathcal{O}_m), K_1(S \otimes \mathcal{O}_n)) \oplus \mathrm{Ext}^1_{\mathbf{Z}}(K_1(S \otimes \mathcal{O}_m), K_0(S \otimes \mathcal{O}_n)) \\ \downarrow \\ KK^0(S \otimes \mathcal{O}_m, S \otimes \mathcal{O}_n) \\ \downarrow \gamma \\ \mathrm{Hom}(K_0(S \otimes \mathcal{O}_m), K_0(S \otimes \mathcal{O}_n)) \oplus \mathrm{Hom}(K_1(S \otimes \mathcal{O}_m), K_1(S \otimes \mathcal{O}_n)) \\ \downarrow \\ 0 \end{array}$$

Let H be the set of classes in $KK^0(S \otimes \mathcal{O}_m, S \otimes \mathcal{O}_n)$ of homomorphisms from $S \otimes \mathcal{O}_m$ to $S \otimes \mathcal{O}_n$. Then H is a subgroup of $KK^0(S \otimes \mathcal{O}_m, S \otimes \mathcal{O}_n)$ for the same reason as in the proof of Lemma 4.10.

We observe that $\gamma(H)$ contains $\mathrm{Hom}(K_0(S \otimes \mathcal{O}_m), K_0(S \otimes \mathcal{O}_n)) \oplus 0$, by considering homomorphisms that factor as

$$S \otimes \mathcal{O}_m \longrightarrow \mathcal{O}_m \longrightarrow \mathcal{O}_n \longrightarrow S \otimes \mathcal{O}_n,$$

where the first map is evaluation at some point of S^1, the second one is taken from Lemma 4.4, and the third one is given by $a \mapsto 1 \otimes a$. Furthermore, if $\alpha \in \mathrm{Hom}(\mathbf{Z}/(m-1)\mathbf{Z}, \mathbf{Z}/(n-1)\mathbf{Z})$, and $\varphi \in \mathrm{Hom}(\mathcal{O}_m, \mathcal{O}_n)$ satisfies $\varphi_* = \alpha$, then $\gamma([\mathrm{id} \otimes \varphi]) = (\alpha, \alpha)$. Since the elements we have exhibited as being in $\gamma(H)$ generate

$$\mathrm{Hom}(K_0(S \otimes \mathcal{O}_m), K_0(S \otimes \mathcal{O}_n)) \oplus \mathrm{Hom}(K_1(S \otimes \mathcal{O}_m), K_1(S \otimes \mathcal{O}_n)),$$

we have shown that $\gamma(H)$ is equal to this entire group. It remains to show that H contains the first term of the exact sequence above.

Let

$$\alpha \in \mathrm{Ext}^1_{\mathbf{Z}}(K_0(S \otimes \mathcal{O}_m), K_1(S \otimes \mathcal{O}_n)) \subset KK^0(S \otimes \mathcal{O}_m, S \otimes \mathcal{O}_n).$$

Let $\varepsilon : S \otimes \mathcal{O}_m \to \mathcal{O}_m$ be evaluation at some point of S^1. Then ε_* is an isomorphism on K_0. Therefore we can form

$$(\varepsilon^*)^{-1}(\alpha) \in \mathrm{Ext}^1_{\mathbf{Z}}(K_0(\mathcal{O}_m), K_1(S \otimes \mathcal{O}_n)) \subset KK^0(\mathcal{O}_m, S \otimes \mathcal{O}_n).$$

Choose by Lemma 4.10 a homomorphism $\varphi : \mathcal{O}_m \to S \otimes \mathcal{O}_n$ such that $[\varphi] = (\varepsilon^*)^{-1}(\alpha)$. Then $\varphi \circ \varepsilon : S \otimes \mathcal{O}_m \to S \otimes \mathcal{O}_n$ satisfies $[\varphi \circ \varepsilon] = \alpha$. So $\mathrm{Ext}^1_{\mathbf{Z}}(K_0(S \otimes \mathcal{O}_m), K_1(S \otimes \mathcal{O}_n)) \subset H$.

A similar argument using Lemma 4.11 and $\iota : \mathcal{O}_n \to S \otimes \mathcal{O}_n$, defined by $\iota(a) = 1 \otimes a$, shows that $\mathrm{Ext}^1_{\mathbf{Z}}(K_1(S \otimes \mathcal{O}_m), K_0(S \otimes \mathcal{O}_n)) \subset H$. This shows $H = KK^0(S \otimes \mathcal{O}_m, S \otimes \mathcal{O}_n)$. ∎

We now prove results in which we impose conditions on the homomorphisms.

4.13 Theorem Let X_1 and X_2 be compact connected subsets of S^1. (Thus, each is either a point, a closed arc, or all of S^1.) Let $m_1, m_2 \geq 2$ be even integers, let n_1, n_2 be integers, and let

$$\alpha \in KK^0(M_{n_1} \otimes C(X_1) \otimes \mathcal{O}_{m_1}, M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2}).$$

Then for every nonzero projection $p \in M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2}$ satisfying

$$[1_{M_{n_1} \otimes C(X_1) \otimes \mathcal{O}_{m_1}}] \times \alpha = [p],$$

there exists a permanently approximately absorbing homomorphism

$$\varphi : M_{n_1} \otimes C(X_1) \otimes \mathcal{O}_{m_1} \to M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2}$$

such that $[\varphi] = \alpha$ and $\varphi(1) = p$.

Proof: Using Lemma 1.11, we can without loss of generality take $p = 1$.

Let e_{11} be a rank one projection in M_{n_1}, and set $e = e_{11} \otimes 1 \otimes 1$. Then $n_1[e] \times \alpha = [1]$ in $K_0(M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2}) \cong K_0(M_{n_2} \otimes \mathcal{O}_{m_2})$. Since $M_{n_2} \otimes \mathcal{O}_{m_2}$ is purely infinite and simple, there exist n_1 Murray-von Neumann equivalent mutually orthogonal projections in $M_{n_2} \otimes \mathcal{O}_{m_2}$ with K_0-classes equal to $[e] \times \alpha$ and which sum to 1. Let p be one of these, and regard p as a projection in $M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2}$. Then

$$M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2} \cong M_{n_1}(p[M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2}]p).$$

It suffices to construct a suitable homomorphism from $C(X_1) \otimes \mathcal{O}_{m_1}$ to $p[M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2}]p$. Thus, without loss of generality, we may assume $n_1 = 1$. Applying Lemma 1.11 again, we reduce again to the case that $p = 1$.

We now construct a homomorphism φ_0 such that $[\varphi_0] = \alpha$, but without requiring that $\varphi_0(1) = 1$ or that φ_0 be permanently approximately absorbing. For this step, it suffices to construct a homomorphism to $C(X_2) \otimes \mathcal{O}_{m_2}$. If X_1 and X_2 are each either a point or S^1, then the existence of the required homomorphism follows from Lemma 4.4, Lemma 4.10, Lemma 4.11, or Theorem 4.12. If X_1 is a closed arc, we compose the map of evaluation at some point of X_1 (which is a homotopy equivalence) with a suitable map obtained from the case in which X_1 is a point. If now X_2 is a closed arc, we compose on the other side with the map from $\mathcal{O}_{m_2}$ to $C(X_2) \otimes \mathcal{O}_{m_2}$ which sends each element of $\mathcal{O}_{m_2}$ to the corresponding constant function. (This map is also a homotopy equivalence.)

We now observe that, just as in the proof of Lemma 1.11, Theorem B of [Zh3] shows that $\varphi_0(1)$ is unitarily equivalent to a constant projection. So we can assume it is a constant projection. If $\varphi_0(1) = 1$, we choose a constant proper isometry $v \in M_{n_2} \otimes C(X_2) \otimes \mathcal{O}_{m_2}$ (constant as a function from X_2 to $M_{n_2} \otimes \mathcal{O}_{m_2}$), and replace φ_0 by $a \mapsto v\varphi_0(a)v^*$. So we can assume $\varphi_0(1)$ is a constant projection different from 1. Let its constant value be f.

Note that in $K_0(M_{n_2} \otimes \mathcal{O}_{m_2})$ we have

$$\begin{aligned} [1-f] &= (\mathrm{ev}_{x_0})_*([1] - [\varphi_0(1)]) \\ &= (\mathrm{ev}_{x_0})_*([1_{C(X_1) \otimes \mathcal{O}_{m_1}}] \times \alpha - [1_{C(X_1) \otimes \mathcal{O}_{m_1}}] \times \alpha) = 0. \end{aligned}$$

Therefore Lemma 1.15 yields a unital permanently approximately absorbing homomorphism

$$\varphi_1 : C(X_1) \otimes \mathcal{O}_{m_1} \to (1-f)(M_{n_2} \otimes \mathcal{O}_{m_2})(1-f)$$

such that $[\varphi_1] = 0$ in $KK^0(C(X_1) \otimes \mathcal{O}_{m_1}, M_{n_2} \otimes \mathcal{O}_{m_2})$. Then we define $\varphi(a) = \varphi_0(a) + 1_{C(X_2)} \otimes \varphi_1(a)$. This is the required homomorphism. ∎

4.14 Lemma Let A be a Cuntz-circle algebra. Then $KK^*(A, -)$ commutes naturally with countable direct limits.

Proof: The algebra A is in the bootstrap category $\mathcal{N}$ of [RS] by Lemma 4.3. Clearly $K_*(A)$ is finitely generated (even finite). Therefore Proposition 7.13 of [RS] implies that $KK^*(A, -)$ is an additive homology theory. The desired result now follows from Section 5 of [Sch2]. ∎

Proof of Theorem 4.1: As in the statement of the theorem, let

$$A = \bigoplus_{i=1}^{r} M_{n(i)} \otimes C(X_i) \otimes \mathcal{O}_{m(i)}$$

be an even Cuntz-circle algebra, let $B = \varinjlim B_k$ be a direct limit of even Cuntz-circle algebras, with maps $\psi_k : B_k \to B$, and let $\alpha \in KK^0(A, B)$. By the definition of a Cuntz-circle algebra, the spaces X_i have only finitely many connected components. Replacing each one by its connected components (and correspondingly increasing the number of summands), we may assume each X_i is connected. It follows from the previous lemma that for every sufficiently large k, there is $\alpha_0 \in KK^0(A, B_k)$ such that $\alpha = \alpha_0 \times [\psi_k]$. If B is unital, then we may assume the maps ψ_k are unital. Choose α_0 as above for some fixed k. Then $[1_A] \times \alpha_0$ and $[1_{B_k}]$ have the same image in $K_0(B)$, and so also have the same image in $K_0(B_l)$ for all sufficiently large l. We replace α_0 by its product with the map from B_k to B_l.

Write $B_k = \bigoplus_{j=1}^{r'} M_{n'(j)} \otimes C(Y_j) \otimes \mathcal{O}_{m'(j)}$, with the Y_j compact connected subsets of S^1. Let $B_k^{(j)}$ be the j-th summand in this expression, and let $A^{(i)}$ be the i-th summand of A. Write $\alpha_0 = \sum_{i,j} \alpha_0^{(i,j)}$ with $\alpha_0^{(i,j)} \in KK^0(A^{(i)}, B_k^{(j)})$. The Künneth formula [Sch1] shows that the map

$$M_{n'(j)} \otimes \mathcal{O}_{m'(j)} \to M_{n'(j)} \otimes C(Y_j) \otimes \mathcal{O}_{m'(j)},$$

given by tensoring with $1_{C(Y_j)}$, is an isomorphism on K_0. Since $M_{n'(j)} \otimes \mathcal{O}_{m'(j)}$ is purely infinite and simple, it follows that we can find nonzero mutually orthogonal projections $p^{(i,j)} \in B_k^{(j)}$ such that $[p^{(i,j)}] = [1_{A^{(i)}}] \times \alpha_0^{(i,j)}$. In the unital case, we have $\sum_i [1_{A^{(i)}}] \times \alpha_0^{(i,j)} = [1_{B_k^{(j)}}]$, and we can require that $\sum_i p^{(i,j)} = 1_{B_k^{(j)}}$. Now use Theorem 4.13 to choose permanently approximately absorbing homomorphisms $\varphi^{(i,j)} : A^{(i)} \to B_k^{(j)}$ such that $[\varphi^{(i,j)}] = \alpha_0^{(i,j)}$. Define $\varphi = \bigoplus_j \sum_i \varphi^{(i,j)}$. Since each $\varphi^{(i,j)}$ is permanently approximately absorbing, so is φ. Also, in the unital case φ is unital. ∎

5 The main results

In this section, we work with algebras in the following class.

5.1 Definition Let $\mathcal{C}$ be the class of simple C^*-algebras A which are direct limits $A \cong \varinjlim A_k$, in which each A_k is an even Cuntz-circle algebra and each map $A_k \to A$ is approximately absorbing.

Our main result (see Theorems 5.4 and 5.17) is that algebras $A \in \mathcal{C}$ are classified up to isomorphism by the K-theoretic invariant $(K_0(A), [1_A], K_1(A))$ in the unital case and $(K_0(A), K_1(A))$ in the nonunital case. (In the first of these expressions, $[1_A]$ is the class in $K_0(A)$ of the identity of A.) The class $\mathcal{C}$ contains all C^*-algebras of the form $B \otimes \mathcal{O}_m$, with m even and B a simple C^*-algebra obtained as a direct limit of finite direct sums of matrix algebras over $C(S^1)$ or $C([0,1])$. In particular, it contains the tensor products of irrational rotation algebras with even Cuntz algebras. These facts give us Corollaries 5.9 through 5.13. The class $\mathcal{C}$ is also closed under the formation of hereditary subalgebras, countable direct limits (provided that the direct limit is simple), and tensor products with simple AF algebras.

In Theorem 5.24, we give a classification theorem for direct limits in which the building blocks are certain simple C^*-algebras, but in which no restriction is made on the maps associated with the direct systems.

If $A \in \mathcal{C}$, then $K_0(A)$ and $K_1(A)$ are countable abelian groups in which every element has finite odd order. We also show in this section that the class $\mathcal{C}$ is large enough that all possible values of $(K_0(A), [1_A], K_1(A))$, with $K_0(A)$ and $K_1(A)$ countable odd torsion groups, are realized by unital algebras $A \in \mathcal{C}$, and that similarly all such values of $(K_0(A), K_1(A))$ are realized by nonunital algebras $A \in \mathcal{C}$. (See Theorems 5.26 and 5.27.)

We begin by establishing our notation for direct limits.

5.2 Notation The notation $A = \varinjlim(A_k, \varphi_{k,k+1})$ will be taken to mean that $(A_k, \varphi_{k,k+1})_{k=1}^{\infty}$ is a direct system of C^*-algebras, with homomorphisms $\varphi_{k,k+1} : A_k \to A_{k+1}$ and direct limit $A = \varinjlim A_k$. We will further implicitly define $\varphi_{k,l} : A_k \to A_l$, for $l \geq k$, to be the composite $\varphi_{l-1,l} \circ \varphi_{l-2,l-1} \circ \cdots \circ \varphi_{k,k+1}$, and $\varphi_{k,\infty} : A_k \to A$ to be the map to the direct limit induced by the system.

A system of finite generating sets for the system $(A_k, \varphi_{k,k+1})_{k=1}^{\infty}$ consists of finite subsets $G_k \subset A_k$ such that G_k generates A_k as a C^*-algebra and $\varphi_{k,k+1}(G_k) \subset G_{k+1}$ for all k. (In most cases of interest, each A_k will be finitely generated, and so such systems will exist.)

We now show that the C^*-algebras in $\mathcal{C}$ are purely infinite.

5.3 Lemma Let $A = \varinjlim(A_k, \varphi_{k,k+1})$ be a direct limit of C^*-algebras A_k, and assume that A is simple. If either

(1) all A_k are even Cuntz-circle algebras, or

(2) all A_k are finite direct sums of purely infinite simple C^*-algebras,

then A is a purely infinite simple C^*-algebra.

Proof: (1) Proposition 7.7 of [Rr1] implies that $\mathcal{O}_m$ is approximately divisible. Therefore each A_k is approximately divisible (see the remark after 1.4 of [BKR]). It is obvious from the definition of approximate divisibility [BKR] that a unital direct limit of approximately divisible C^*-algebras is approximately divisible. If A is unital, it therefore follows that A is approximately divisible. Clearly A is infinite, so it is purely infinite by Theorem 1.4 (a) of [BKR].

If A is not unital, choose k and an infinite projection $p \in A_k$ such that $\varphi_{k,\infty}(p) \neq 0$. Corollary 2.9 of [BKR] implies that $\varphi_{k,l}(p)A_l\varphi_{k,l}(p)$ is approximately divisible for $l \geq k$. Therefore $\varphi_{k,\infty}(p)A\varphi_{k,\infty}(p) = \varinjlim \varphi_{k,l}(p)A_l\varphi_{k,l}(p)$ is approximately divisible and infinite simple, hence purely infinite. Now A is stably isomorphic to $\varphi_{k,\infty}(p)A\varphi_{k,\infty}(p)$ (see [Bn1]), and hence also purely infinite.

(2) We may assume that each $\varphi_{k,\infty}$ is injective. (If not, we replace A_k by $A_k/\ker(\varphi_{k,\infty})$, which is again a finite direct sum of purely infinite simple C^*-algebras.)

By Theorem 1.2 (i) of [Zh1], every purely infinite simple C^*-algebra has real rank zero. It clearly follows from 3.1 of [BP] that A has real rank zero. Therefore it suffices to show that every nonzero projection $p \in A$ is infinite. Choose k and a projection $q \in A_k$ such that $\varphi_{k,\infty}(q)$ is unitarily equivalent to p. It suffices to prove that that $\varphi_{k,\infty}(q)$ is infinite. But it is immediate that q is infinite in A_k, and infiniteness of $\varphi_{k,\infty}(q)$ now follows from injectivity of $\varphi_{k,\infty}$. ∎

The following result is our main theorem, from which most of the other results in this section will follow.

5.4 Theorem Let $A = \varinjlim (A_n, \varphi_{n,n+1})$ and $B = \varinjlim (B_n, \psi_{n,n+1})$ be two simple C^*-algebras which are direct limits of even Cuntz-circle algebras, and assume that the maps $\varphi_{n,\infty}$ and $\psi_{n,\infty}$ are all unital and approximately absorbing. If

$$(K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B)),$$

then $A \cong B$.

The proof consists of constructing an approximate intertwining of the direct systems, as was first done in [Ell2]. We will use the particular statement given in [Thn].

If all the maps of the system were injective, this construction would be fairly direct from Theorems 3.7 and 4.1. Lack of injectivity causes some technical problems; in particular, we must replace the system $(A_k, \varphi_{k,k+1})_{k=1}^{\infty}$ by a system $(A'_k, \varphi'_{k,k+1})_{k=1}^{\infty}$ in which the maps $\varphi'_{k,\infty}$ are approximately injective (see Definition 3.8), and similarly with the system $(B_k, \psi_{k,k+1})_{k=1}^{\infty}$. We have to construct the A'_k and B'_k at the same time as the approximate intertwining, which makes the proof somewhat complicated. We will isolate the actual computational steps as the following two lemmas; the proof of the theorem will then consist mainly of keeping track of many maps and indices.

In the first of these lemmas, we refer to compact subsets of S^1 with finitely many components. Note that such a subset is either S^1 itself or a finite disjoint union of closed arcs and points.

5.5 Lemma Let $A = \varinjlim(A_k, \varphi_{k,k+1})$ be a simple direct limit of even Cuntz-circle algebras. Thus, in particular we can write $A_1 = \bigoplus_{i=1}^{r} C(X_i) \otimes D_i$, where the D_i are matrix algebras over even Cuntz algebras and the X_i are compact subsets of S^1 with finitely many components. Assume that each $\varphi_{k,k+1}$ is unital and that $\varphi_{1,\infty}$ is approximately absorbing. Let $G \subset A_1$ be finite, and let $\varepsilon > 0$.

Then there exist compact subsets X'_i of X_i with finitely many components such that the C^*-algebra $A'_1 = \bigoplus_{i=1}^{r} C(X'_i) \otimes D_i$ and the restriction map $\alpha : A_1 \to A'_1$ satisfy the following. There is $l > 1$ and a homomorphism $\alpha' : A'_1 \to A_l$ such that $\alpha' \circ \alpha = \varphi_{1,l}$, the map $\varphi_{l,\infty} \circ \alpha'$ is ε-approximately injective with respect to the finite set $\alpha(G)$ (see Definition 3.8), and $\varphi_{l,\infty} \circ \alpha'$ is approximately absorbing.

Proof: For simplicity of notation, we will assume that $A_1 = C(X, D)$, with $X \subset S^1$ and D simple. (Thus, we are assuming A_1 has only one summand. As will be clear from the proof, if it has more, we will be able to use the largest of the values of l associated to the summands.) Choose $\delta > 0$ such that whenever $x_1, x_2 \in X$ satisfy $|x_1 - x_2| \leq \delta$ and $f \in G$, then $\|f(x_1) - f(x_2)\| < \varepsilon$. Let $u \otimes 1 \in C(X, D)$ be the usual standard

generator of $C(S^1) \otimes \mathbf{C}$, and note that

$$\mathrm{sp}(\varphi_{1,\infty}(u \otimes 1)) = \bigcap_{l=1}^{\infty} \mathrm{sp}(\varphi_{1,l}(u \otimes 1)).$$

A standard compactness argument yields $l \geq 1$ such that $\mathrm{sp}(\varphi_{1,l}(u \otimes 1))$ is contained in a $\delta/2$-neighborhood of $\mathrm{sp}(\varphi_{1,\infty}(u \otimes 1))$. Furthermore, the complement $X \setminus \mathrm{sp}(\varphi_{1,l}(u \otimes 1))$ is a countable disjoint union of arcs open in X, so repeating the compactness argument gives $W \subset X$, a finite union of arcs open in X, such that $X' = X \setminus W$ is contained in a $\delta/2$-neighborhood of $\mathrm{sp}(\varphi_{1,l}(u \otimes 1))$.

Define $A_1' = C(X', D)$, and let α be the restriction map. Note that

$$\ker(\alpha) = C_0(X \setminus X', D) \subset C_0(X \setminus \mathrm{sp}(\varphi_{1,l}(u \otimes 1)), D) = \ker(\varphi_{1,l}).$$

Therefore $\varphi_{1,l}$ factors through α; let $\alpha' : A_1' \to A_l$ be the resulting map. Then

$$\ker(\varphi_{l,\infty} \circ \alpha') = C_0(X' \setminus \mathrm{sp}(\varphi_{1,\infty}(u \otimes 1)), D),$$

and X' is contained in a δ-neighborhood of $\mathrm{sp}(\varphi_{1,\infty}(u \otimes 1))$, so the choice of δ ensures that $\varphi_{l,\infty} \circ \alpha'$ is ε-approximately injective.

It remains to prove that $\varphi_{l,\infty} \circ \alpha'$ is approximately absorbing. Since $\varphi_{1,\infty}$ is approximately absorbing, it is easy to check this directly from the definition, using the relation $(\varphi_{l,\infty} \circ \alpha') \circ \alpha = \varphi_{1,\infty}$ and the fact that α is a restriction map. ∎

5.6 Notation Let A and B be C^*-algebras, and let $\varphi, \psi : A \to B$ be homomorphisms. If $G \subset A$ and $\varepsilon > 0$, then we write $\varphi \overset{\varepsilon}{\approx} \psi$ (with respect to G) to mean $\|\varphi(a) - \psi(a)\| < \varepsilon$ for all $a \in G$. (Compare with the relation $\varphi \overset{\varepsilon}{\sim} \psi$ in Definition 1.1.)

5.7 Lemma Let $A = \varinjlim(A_k, \varphi_{k,k+1})$ be a simple direct limit of even Cuntz-circle algebras, with all $\varphi_{k,k+1}$ unital, let D be an even Cuntz-circle algebra, and let $\rho : A_1 \to D$ and $\theta_0 : D \to A_2$ be unital homomorphisms. Let $0 < \varepsilon \leq 1/2$, and let $F \subset A_1$ be a finite generating set. Assume that

$$[\rho] \times [\theta_0] \times [\varphi_{2,\infty}] = [\varphi_{1,\infty}] \quad \text{in } KK^0(A_1, A),$$

that $\varphi_{2,\infty} \circ \theta_0 \circ \rho$ is injective and approximately absorbing, that θ_0 is permanently approximately absorbing, and that $\varphi_{1,\infty}$ is approximately absorbing and ε-approximately injective with respect F. Finally, assume that, with respect to some realization of A_1 as a finite direct sum as in the definition of a Cuntz-circle algebra, the set F contains the identities of all the summands.

Then there is $l \geq 2$ and a unital permanently approximately absorbing homomorphism $\theta : D \to A_l$ such that $[\theta] = [\theta_0] \times [\varphi_{2,l}]$ in $KK^0(D, A_l)$, such that $[\rho] \times [\theta] = [\varphi_{1,l}]$ in $KK^0(A_1, A_l)$, and such that $\theta \circ \rho \overset{4\varepsilon}{\approx} \varphi_{1,l}$ with respect to F.

Proof: Without loss of generality, assume the elements of F all have norm at most 1.

Since $KK^0(A_1, -)$ commutes with countable direct limits (by Lemma 4.14), there is $m \geq 2$ such that

$$[\rho] \times [\theta_0] \times [\varphi_{2,m}] = [\varphi_{1,m}] \quad \text{in } KK^0(A_1, A_m).$$

Also, the hypotheses and Theorem 3.11 imply that $\varphi_{2,\infty} \circ \theta_0 \circ \rho \overset{2\varepsilon}{\sim} \varphi_{1,\infty}$ with respect to F. (We have A purely infinite by Lemma 5.3 (1). Also, we apply Theorem 3.11 to each summand separately. That is, if $A_1 = A_{11} \oplus \cdots \oplus A_{1r}$, and e_i is the identity of A_{1i}, then we first observe that $(\varphi_{2,\infty} \circ \theta_0 \circ \rho)(e_i)$ is close to $\varphi_{1,\infty}(e_i)$. Thus these projections are unitarily equivalent for each i; without loss of generality, we assume

they are equal for each i. Now use Theorem 3.11 on the restrictions of $\varphi_{2,\infty} \circ \theta_0 \circ \rho$ and $\varphi_{1,\infty}$ to each A_{1i}, regarded as maps to $\varphi_{1,\infty}(e_i)A\varphi_{1,\infty}(e_i)$.)

There is thus a unitary $v \in A$ such that

$$\|v^*\varphi_{1,\infty}(a)v - (\varphi_{2,\infty} \circ \theta_0 \circ \rho)(a)\| < 2\varepsilon$$

for all $a \in F$. Choose $m' \geq m$ such that there is a unitary $w \in A_{m'}$ with $\|\varphi_{m',\infty}(w) - v\| < \varepsilon/2$. Then for each $a \in F$, we have

$$\|\varphi_{m',\infty}\left(w^*\varphi_{1,m'}(a)w - (\varphi_{2,m'} \circ \theta_0 \circ \rho)(a)\right)\| < 3\varepsilon.$$

Since F is finite, there is therefore $l \geq m'$ such that

$$\|\varphi_{m',l}\left(w^*\varphi_{1,m'}(a)w - (\varphi_{2,m'} \circ \theta_0 \circ \rho)(a)\right)\| < 4\varepsilon$$

for all $a \in F$.

Let $z = \varphi_{m',l}(w)$, and define $\theta(b) = z(\varphi_{2,l} \circ \theta_0)(b)z^*$ for $b \in D$. Then $\theta \circ \rho \overset{4\varepsilon}{\approx} \varphi_{1,l}$ with respect to F. Note that θ is the composite of a unital homomorphism with the permanently approximately absorbing homomorphism θ_0, and hence still permanently approximately absorbing by Lemma 1.14. Also, conjugation by a unitary does not change the class in KK-theory. Therefore $[\theta] = [\theta_0] \times [\varphi_{2,l}]$. Furthermore,

$$[\rho] \times [\theta] = [\rho] \times [\theta_0] \times [\varphi_{2,m}] \times [\varphi_{m,l}] = [\varphi_{1,l}]$$

in $KK^0(A_1, A)$, by the choice of m at the beginning of the proof. ∎

Proof of Theorem 5.4: Since $K_i(A) \cong K_i(B)$, the Universal Coefficient Theorem (Theorem 1.17 of [RS]) and Proposition 7.3 of [RS] yield an invertible $\sigma \in KK^0(A, B)$. (Lemma 4.3 implies that both A and B are in the bootstrap category $\mathcal{N}$ of [RS].) Let $\sigma^{-1} \in KK^0(B, A)$ be the inverse.

Fix realizations of each A_k and B_k as direct sums as in the definition of a Cuntz-circle algebra. (Quotients of A_k and B_k will then be realized as direct sums in the same way.) Let $(F_k)_{k=1}^{\infty}$ and $(G_k)_{k=1}^{\infty}$ be systems of finite generating sets for $(A_k, \varphi_{k,k+1})_{k=1}^{\infty}$ and $(B_k, \psi_{k,k+1})_{k=1}^{\infty}$ respectively. We require that F_k contain the identities of the summands of A_k, and similarly for G_k and B_k. We construct a diagram as follows:

$$\begin{tikzcd}
A_{m(1)} \arrow[r, "\varphi_{m(1),m(2)}"] \arrow[d, "\alpha_1"'] & A_{m(2)} \arrow[r, "\varphi_{m(2),m(3)}"] \arrow[d, "\alpha_2"'] & A_{m(3)} \arrow[r, "\varphi_{m(3),m(4)}"] \arrow[d, "\alpha_3"'] & \cdots \\
A'_1 \arrow[r, "\varphi'_{1,2}"] \arrow[ur, "\alpha'_1"] \arrow[d, "\rho_1"'] & A'_2 \arrow[r, "\varphi'_{2,3}"] \arrow[ur, "\alpha'_2"] \arrow[d, "\rho_2"'] & A'_3 \arrow[r, "\varphi'_{3,4}"] \arrow[ur, "\alpha'_3"] \arrow[d, "\rho_3"'] & \cdots \\
B'_1 \arrow[r, "\psi'_{1,2}"] \arrow[ur, "\theta_1"] \arrow[dr, "\beta'_1"] & B'_2 \arrow[r, "\psi'_{2,3}"] \arrow[ur, "\theta_2"] \arrow[dr, "\beta'_2"] & B'_3 \arrow[r, "\psi'_{3,4}"] \arrow[ur, "\theta_3"] \arrow[dr, "\beta'_3"] & \cdots \\
B_{n(1)} \arrow[r, "\psi_{n(1),n(2)}"'] \arrow[u, "\beta_1"'] & B_{n(2)} \arrow[r, "\psi_{n(2),n(3)}"'] \arrow[u, "\beta_2"'] & B_{n(3)} \arrow[r, "\psi_{n(3),n(4)}"'] \arrow[u, "\beta_3"'] & \cdots
\end{tikzcd}$$

Here, all triangles in the top and bottom sections are supposed to commute. In the middle section, we require that $\theta_k \circ \rho_k \overset{1/2^k}{\approx} \varphi'_{k,k+1}$ with respect to $\alpha_k(F_{m(k)}) \cup (\theta_{k-1} \circ \beta_{k-1})(G_{n(k-1)})$, and $\rho_{k+1} \circ \theta_k \overset{1/2^k}{\approx} \psi'_{k,k+1}$

with respect to $\beta_k(G_{n(k)}) \cup (\rho_k \circ \alpha_k)(F_{m(k)})$. In order to make this happen, we will also construct systems $(F'_k)_{k=1}^{\infty}$ and $(G'_k)_{k=1}^{\infty}$ of finite generating sets for $(A'_k, \varphi'_{k,k+1})_{k=1}^{\infty}$ and $(B'_k, \psi'_{k,k+1})_{k=1}^{\infty}$ respectively, such that

$$\alpha_k(F_{m(k)}) \cup (\theta_{k-1} \circ \beta_{k-1})(G_{n(k-1)}) \subset F'_k \text{ and } \beta_k(G_{n(k)}) \cup (\rho_k \circ \alpha_k)(F_{m(k)}) \subset G'_k.$$

We will then require that θ_k and ρ_k be permanently approximately absorbing, that $\varphi'_{k,\infty}$ be approximately absorbing and $(1/2^{k+2})$-approximately injective with respect to F'_k, and that $\psi'_{k,\infty}$ be approximately absorbing and $(1/2^{k+2})$-approximately injective with respect to G'_k.

We construct this diagram by induction on the column number. Some of the steps will start by going further out in one of the original direct systems, and so we will have to construct temporary versions of some of the maps in the diagram. They will be distinguished from the final ones with tildes.

We show the details only for columns 1 and 2; the remaining steps are essentially the same as for column 2.

Step 1, part A: Set $m(1) = 1$. Use Lemma 5.5 to construct A'_1, a surjective homomorphism $\alpha_1 : A_{m(1)} \to A'_1$, an integer $i(1) > m(1)$, and a homomorphism $\tilde{\alpha}'_1 : A'_1 \to A_{i(1)}$ such that $\tilde{\alpha}'_1 \circ \alpha_1 = \varphi_{1,i(1)}$, and the map $\varphi_{i(1),\infty} \circ \tilde{\alpha}'_1$ is 1/8-approximately injective with respect to the finite set $\alpha_1(F_1)$ and is approximately absorbing. Define $F'_1 = \alpha_1(F_1)$. Since α_1 is a direct sum of restriction maps, and F_1 contains the identities of the summands of A_1, the set F'_1 contains the identities of the summands of A'_1.

Step 1, part B: Use Theorem 4.1 to find $n(1)$ and a permanently approximately absorbing unital homomorphism $\tilde{\rho}_1 : A'_1 \to B_{n(1)}$ such that

$$[\tilde{\rho}_1] \times [\psi_{n(1),\infty}] = [\tilde{\alpha}'_1] \times [\varphi_{i(1),\infty}] \times \sigma \quad \text{in } KK^0(A'_1, B).$$

Use Lemma 5.5 to find B'_1, a homomorphism $\beta_1 : B_{n(1)} \to B'_1$, an integer $j(1) > n(1)$, and a homomorphism $\tilde{\beta}'_1 : B'_1 \to B_{j(1)}$ such that $\tilde{\beta}'_1 \circ \beta_1 = \psi_{n(1),j(1)}$, and the map $\varphi_{j(1),\infty} \circ \tilde{\beta}'_1$ is 1/8-approximately injective with respect to the finite set $\beta_1(G_{n(1)} \cup \tilde{\rho}_1(F'_1))$ and is approximately absorbing.

Define $G'_1 = \beta_1(G_{n(1)} \cup \tilde{\rho}_1(F'_1))$. Then G'_1 contains the identities of the summands of B'_1. Further define $\rho_1 = \beta_1 \circ \tilde{\rho}_1$. Notice that the surjectivity of β_1 implies that ρ_1 is still permanently approximately absorbing. Furthermore, we have $\tilde{\beta}'_1 \circ \rho_1 = \psi_{n(1),j(1)} \circ \tilde{\rho}_1$, whence

$$[\rho_1] \times [\tilde{\beta}_1] \times [\psi_{j(1),\infty}] = [\tilde{\alpha}'_1] \times [\varphi_{i(1),\infty}] \times \sigma \quad \text{in } KK^0(A'_1, B).$$

Step 2, part A: Use Theorem 4.1 to find $l \geq i(1)$, where $i(1)$ is as chosen in Step 1 part A, and a permanently approximately absorbing unital homomorphism $\tilde{\theta}_1 : B'_1 \to A_l$ such that

$$[\tilde{\theta}_1] \times [\varphi_{l,\infty}] = [\tilde{\beta}'_1] \times [\psi_{j(1),\infty}] \times \sigma^{-1} \quad \text{in } KK^0(B'_1, A).$$

Now compute:

$$\begin{aligned}[\tilde{\alpha}'_1] \times [\varphi_{i(1),l}] \times [\varphi_{l,\infty}] &= [\tilde{\alpha}'_1] \times [\varphi_{i(1),\infty}] \times \sigma \times \sigma^{-1} \\ &= [\rho_1] \times [\tilde{\beta}'_1] \times [\psi_{j(1),\infty}] \times \sigma^{-1} = [\rho_1] \times [\tilde{\theta}_1] \times [\varphi_{l,\infty}]\end{aligned}$$

in $KK^0(A'_1, A)$. We next apply Lemma 5.7 with $D = B'_1$, $\rho = \rho_1$, $\theta_0 = \tilde{\theta}_1$, and the direct system being

$$A'_1 \xrightarrow{\varphi_{i(1),l} \circ \tilde{\alpha}'_1} A_l \to A_{l+1} \to A_{l+2} \to \cdots.$$

Note that $\varphi_{l,\infty} \circ \varphi_{i(1),l} \circ \tilde{\alpha}'_1$ is 1/8-approximately injective with respect to F'_1, by Step 1 part A, and that $\varphi_{l,\infty} \circ \tilde{\theta}_1 \circ \rho_1$ is injective and approximately absorbing by Lemma 1.14 (because ρ_1 is permanently approximately absorbing and $\varphi_{l,\infty} \circ \tilde{\theta}_1$ is unital). Therefore Lemma 5.7 yields $m(2) \geq l$ and a unital

permanently approximately absorbing homomorphism $\tilde{\tilde{\theta}}_1 : B_1' \to A_{m(2)}$ such that $[\tilde{\tilde{\theta}}_1] = [\tilde{\theta}_1] \times [\varphi_{l,m(2)}]$ and $[\rho_1] \times [\tilde{\tilde{\theta}}_1] = [\tilde{\alpha}_1'] \times [\varphi_{i(1),m(2)}]$, and also $\tilde{\tilde{\theta}}_1 \circ \rho_1 \overset{1/2}{\approx} \varphi_{i(1),m(2)} \circ \tilde{\alpha}_1'$ with respect to F_1'.

Define $\alpha_1' = \varphi_{i(1),m(2)} \circ \tilde{\alpha}_1'$.

Use Lemma 5.5 as before to construct A_2', a surjective homomorphism $\alpha_2 : A_{m(2)} \to A_2'$, an integer $i(2) > m(2)$, and a homomorphism $\tilde{\alpha}_2' : A_2' \to A_{i(2)}$ such that $\tilde{\alpha}_2' \circ \alpha_2 = \varphi_{m(2),i(2)}$, the map $\varphi_{i(2),\infty} \circ \tilde{\alpha}_2'$ is 1/16-approximately injective with respect to $\alpha_2(F_{m(2)} \cup \tilde{\tilde{\theta}}_1(G_1'))$, and this map is approximately absorbing. Define $\theta_1 = \alpha_2 \circ \tilde{\tilde{\theta}}_1$, which is still permanently approximately absorbing since α_2 is surjective. Define $\varphi_{1,2}' = \alpha_2 \circ \alpha_1'$, and define $F_2' = \alpha_2(F_{m(2)} \cup \tilde{\tilde{\theta}}_1(F_1'))$.

Note that $\theta_1 \circ \rho_1 = \alpha_2 \circ \tilde{\tilde{\theta}}_1 \circ \rho_1$ and $\varphi_{1,2}' = \alpha_2 \circ \varphi_{i(1),m(1)} \circ \tilde{\alpha}_1'$. It follows that $\theta_1 \circ \rho_1 \overset{1/2}{\approx} \varphi_{1,2}'$ and $[\rho_1] \times [\theta_1] = [\varphi_{1,2}']$. Furthermore,

$$\begin{aligned}
[\theta_1] \times [\tilde{\alpha}_2'] \times [\varphi_{i(2),\infty}] &= [\tilde{\tilde{\theta}}_1] \times [\varphi_{m(2),\infty}] \\
&= [\tilde{\theta}_1] \times [\varphi_{l,\infty}] = [\tilde{\beta}_1'] \times [\psi_{j(1),\infty}] \times \sigma^{-1}
\end{aligned}$$

in $KK^0(B_1', A)$. Finally, F_2' contains the identities of the summands of A_2'.

Step 2 part B: This step is similar to part A, so we will be briefer. Use Theorem 4.1 to find $l \geq j(1)$ and a permanently approximately absorbing unital homomorphism $\tilde{\rho}_2 : A_2' \to B_l$ such that $[\tilde{\rho}_2] \times [\psi_{l,\infty}] = [\tilde{\alpha}_2'] \times [\varphi_{i(2),\infty}] \times \sigma$. Use Lemma 5.7 to find $n(2) \geq l$ and a unital permanently approximately absorbing homomorphism $\tilde{\tilde{\rho}}_2 : A_2' \to B_{n(2)}$ such that $[\tilde{\tilde{\rho}}_2] = [\tilde{\rho}_2] \times [\psi_{l,n(2)}]$ and $[\theta_1] \times [\tilde{\tilde{\rho}}_2] = [\tilde{\beta}_1'] \times [\psi_{j(1),n(2)}]$, and also $\tilde{\tilde{\rho}}_2 \circ \theta_1 \overset{1/2}{\approx} \psi_{j(1),n(2)} \circ \tilde{\beta}_1'$ with respect to G_1'. Define $\beta_1' = \psi_{j(1),n(2)} \circ \tilde{\beta}_1'$.

Now use Lemma 5.5 to produce a surjective homomorphism $\beta_2 : B_{n(2)} \to B_2'$ and a homomorphism $\tilde{\beta}_2' : B_2' \to B_{j(2)}$ (with $j(2) > n(2)$) such that $\tilde{\beta}_2' \circ \beta_2 = \psi_{n(2),j(2)}$, the map $\psi_{j(2),\infty} \circ \tilde{\beta}_2'$ is 1/16-approximately injective with respect to $\beta_2(G_{n(2)} \cup \tilde{\tilde{\rho}}_2(F_2'))$, and $\psi_{j(2),\infty} \circ \tilde{\beta}_2'$ is approximately absorbing. Define $\rho_2 = \beta_2 \circ \tilde{\tilde{\rho}}_2$ and $\psi_{1,2}' = \beta_2 \circ \beta_1'$, and set $G_2' = \beta_2(G_{n(2)} \cup \tilde{\tilde{\rho}}_2(F_2'))$. Then check that $\rho_2 \circ \theta_1 \overset{1/2}{\approx} \psi_{1,2}'$, that $[\theta_1] \times [\rho_2] = [\psi_{1,2}']$, that $[\rho_2] \times [\tilde{\beta}_2'] \times [\psi_{j(2),\infty}] = [\tilde{\alpha}_2'] \times [\varphi_{i(2),\infty}] \times \sigma$, and that G_2' contains the identities of the summands of B_2'. This completes part B of Step 2.

Each successive half step consists of a repetition of the argument already used in the two halves of Step 2, using, in order, Theorem 4.1, Lemma 5.7, and Lemma 5.5. In Step k part A, we choose $\varphi_{i(k),\infty} \circ \tilde{\alpha}_k'$ to be $(1/2^{k+2})$-approximately injective, and we get $\theta_{k-1} \circ \rho_{k-1} \overset{1/2^{k-1}}{\approx} \varphi_{k-1,k}'$ with respect to F_{k-1}'. In part B, we choose $\psi_{j(k),\infty} \circ \tilde{\beta}_k'$ to be $(1/2^{k+2})$-approximately injective, and we get $\rho_k \circ \theta_{k-1} \overset{1/2^{k-1}}{\approx} \psi_{k-1,k}'$ with respect to G_{k-1}'.

With this construction, the top and bottom rows of triangles in our diagram commute. Therefore they induce isomorphisms $\varinjlim A_k \cong \varinjlim A_k'$ and $\varinjlim B_k \cong \varinjlim B_k'$ The middle row of triangles is an approximate intertwining in the sense of Definition 2 of [Thn]. By Theorem 3 of [Thn], it therefore induces an isomorphism $\varinjlim A_k' \cong \varinjlim B_k'$. So $A \cong B$. ∎

One easily checks that the isomorphism constructed in this proof induces the same map on K-theory as σ. However, it is not clear that its class in $KK^0(A, B)$ is equal to σ.

A one sided version of the previous proof, using Lemma 1 of [Thn], establishes the following:

5.8 Proposition Let A and B as in Theorem 5.4. If there are homomorphisms

$$\alpha_0 : K_0(A) \to K_0(B) \quad \text{and} \quad \alpha_1 : K_1(A) \to K_1(B)$$

such that $\alpha_0([1_A]) = [1_B]$, then there is a unital homomorphism $\varphi : A \to B$ such that

$$\varphi_*^{(0)} = \alpha_0 \quad \text{and} \quad \varphi_*^{(1)} = \alpha_1. \ \blacksquare$$

We now give a number of corollaries of Theorem 5.4.

5.9 Corollary Let $A = \varinjlim (A_k, \varphi_{k,k+1})$ and $B = \varinjlim (B_k, \psi_{k,k+1})$ be unital simple C^*-algebras, such that each A_k and each B_k is a finite direct sum of matrix algebras over algebras $C(X)$, with each X being a point, a compact interval, or a circle. (A and B can independently have real rank either 0 or 1.) If m is even and

$$(K_0(A \otimes \mathcal{O}_m), [1_{A \otimes \mathcal{O}_m}], K_1(A \otimes \mathcal{O}_m)) \cong (K_0(B \otimes \mathcal{O}_m), [1_{B \otimes \mathcal{O}_m}], K_1(B \otimes \mathcal{O}_m)),$$

then

$$A \otimes \mathcal{O}_m \cong B \otimes \mathcal{O}_m.$$

Proof: We may assume that the $\varphi_{k,k+1}$ and $\psi_{k,k+1}$ are unital. Let $\Phi_{k,k+1} = \varphi_{k,k+1} \otimes \mathrm{id}_{\mathcal{O}_m}$ and $\Psi_{k,k+1} = \psi_{k,k+1} \otimes \mathrm{id}_{\mathcal{O}_m}$ Then

$$A \otimes \mathcal{O}_m \cong \varinjlim (A_k \otimes \mathcal{O}_m, \Phi_{k,k+1}) \quad \text{and} \quad B \otimes \mathcal{O}_m \cong \varinjlim (B_k \otimes \mathcal{O}_m, \Psi_{k,k+1}).$$

Both $A \otimes \mathcal{O}_m$ and $B \otimes \mathcal{O}_m$ are simple, hence (by Lemma 5.3 (1)) both are purely infinite simple. It follows from Corollary 1.9 that $\Phi_{k,\infty}$ and $\Psi_{k,\infty}$ are approximately absorbing. Therefore Theorem 5.4 applies. $\blacksquare$

5.10 Corollary Let A and B be simple direct limits of circle algebras etc. as in Corollary 5.9, and let m be even. If

$$(K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B))$$

(ignoring order), then

$$A \otimes \mathcal{O}_m \cong B \otimes \mathcal{O}_m.$$

Proof: The Künneth formula [Sch1] and its splitting (see Remark 7.11 of [RS]) imply that

$$(K_0(A \otimes \mathcal{O}_m), [1_{A \otimes \mathcal{O}_m}], K_1(A \otimes \mathcal{O}_m)) \cong (K_0(B \otimes \mathcal{O}_m), [1_{B \otimes \mathcal{O}_m}], K_1(B \otimes \mathcal{O}_m)).$$

$\blacksquare$

5.11 Corollary Let $A = \varinjlim (A_k, \varphi_{k,k+1})$ and $B = \varinjlim (B_k, \psi_{k,k+1})$ be unital simple C^*-algebras, such that each A_k and each B_k is a finite direct sum of matrix algebras over algebras $C(X)$, with each X being a compact subset of S^1. (A and B can independently have real rank either 0 or 1.) If m is even and

$$(K_0(A \otimes \mathcal{O}_m), [1_{A \otimes \mathcal{O}_m}], K_1(A \otimes \mathcal{O}_m)) \cong (K_0(B \otimes \mathcal{O}_m), [1_{B \otimes \mathcal{O}_m}], K_1(B \otimes \mathcal{O}_m)),$$

then

$$A \otimes \mathcal{O}_m \cong B \otimes \mathcal{O}_m.$$

Proof: We only need to show that A and B can be rewritten as direct limits as above, but with the restriction that the subsets $X \subset S^1$ have only finitely many components. Now A and B clearly satisfy condition (ii) of

Theorem 4.3 of [Ell2]. That theorem therefore implies they are direct limits of finite direct sums of matrix algebras over $C(S^1)$. ∎

5.12 Corollary Let A_{θ_1} and A_{θ_2} be two irrational rotation algebras and let m be even. Then

$$A_{\theta_1} \otimes \mathcal{O}_m \cong A_{\theta_2} \otimes \mathcal{O}_m.$$

Proof: It follows from [EE] that every irrational rotation algebra is a simple (unital) direct limit of finite direct sums of matrix algebras over $C(S^1)$. The Pimsner-Voiculescu exact sequence [PV2] shows that, ignoring the order, $K_0(A_\theta) \cong K_1(A_\theta) \cong \mathbf{Z} \oplus \mathbf{Z}$ for all θ. ∎

By contrast, recall that by [Rf] and [PV1], we have $A_{\theta_1} \cong A_{\theta_2}$ only when $\theta_1 = \pm\theta_2 \pmod{\mathbf{Z}}$.

5.13 Corollary (Corollary 3.6 of [Ln3]) Let A be a simple direct limit of circle algebras etc., as in Corollary 5.9. Then $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. ∎

We now prove that the class $\boldsymbol{\mathcal{C}}$ is closed under several natural operations. These results will enable us to extend the classification results above to the nonunital case.

5.14 Lemma If $A \in \boldsymbol{\mathcal{C}}$, then $A \otimes \mathcal{K} \in \boldsymbol{\mathcal{C}}$.

Proof: Suppose that $A = \varinjlim (A_k, \varphi_{k,k+1})$, where the A_k are even Cuntz-circle algebras and the $\varphi_{k,k+1}$ are approximately absorbing. Define $\psi_{k,k+1} : M_k(A_k) \to M_{k+1}(A_{k+1})$ by $\psi_{k,k+1} = \mathrm{id}_{M_k} \otimes \varphi_{k,k+1} \oplus 0$. Then

$$\varinjlim (M_k(A_k), \psi_{k,k+1}) \cong A \otimes \mathcal{K}.$$

Note that $\psi_{k,\infty}$ maps $M_k(A_k)$ into $M_k(A)$. Corollary 1.10 implies that $\psi_{k,\infty}$ is approximately absorbing. Therefore $A \otimes \mathcal{K} \in \boldsymbol{\mathcal{C}}$. ∎

5.15 Lemma Let A be a simple C^*-algebra in $\boldsymbol{\mathcal{C}}$, and let $p \in A$ be a nonzero projection. Then $pAp \in \boldsymbol{\mathcal{C}}$.

Proof: Write $A = \varinjlim (A_k, \varphi_{k,k+1})$, where each A_k is an even Cuntz-circle algebra and each $\varphi_{k,\infty}$ is approximately absorbing. There is l and a projection $q \in A_l$ such that $\|\varphi_{l,\infty}(q) - p\| < 1$. Therefore $\varphi_{l,\infty}(q)$ is unitarily equivalent to p. It follows that

$$pAp \cong \varphi_{l,\infty}(q) A \varphi_{l,\infty}(q) \cong \varinjlim_{k \geq l} \varphi_{l,k}(q) A_k \varphi_{l,k}(q).$$

Lemma 1.11 implies that the algebras in this direct system are even Cuntz-circle algebras, and Lemma 1.12 implies that the maps from them to the direct limit are approximately absorbing. Certainly pAp is simple, so $pAp \in \boldsymbol{\mathcal{C}}$. ∎

5.16 Corollary Let $A \in \boldsymbol{\mathcal{C}}$ and let B be a hereditary C^*-subalgebra of A. Then $B \in \boldsymbol{\mathcal{C}}$.

Proof: If B is unital, then $B = pAp$ for some projection $p \in A$. So $B \in \boldsymbol{\mathcal{C}}$ by Lemma 5.15. Otherwise, note that A is purely infinite by Lemma 5.3 (1). Therefore B is stable by Theorem 1.2 (i) of [Zh1]. Using [Bn1], it follows that $B \cong A \otimes \mathcal{K}$, whence $B \in \boldsymbol{\mathcal{C}}$ by Lemma 5.14. ∎

5.17 Theorem Let A and B be two simple C^*-algebras in $\boldsymbol{\mathcal{C}}$. Suppose that both A and B are nonunital and that

$$(K_0(A), K_1(A)) \cong (K_0(B), K_1(B)).$$

Then $A \cong B$.

Proof: By Lemma 5.3 and Theorem 1.2 (i) of [Zh1], both A and B are stable. Let $p \in A$ be a nonzero projection. Then [Bn1] implies that $A \cong pAp \otimes \mathcal{K}$. Let $\alpha : K_0(A) \to K_0(B)$ be an isomorphism. Then there is a nonzero projection $q \in B$ such that $\alpha([p]) = [q]$. By [Bn1] again, $B \cong qBq \otimes \mathcal{K}$. It follows from Lemma 5.15 that $pAp,\, qBq \in \boldsymbol{\mathcal{C}}$. Theorem 5.4 therefore gives $pAp \cong qBq$. Consequently

$$A \cong pAp \otimes \mathcal{K} \cong qBq \otimes \mathcal{K} \cong B. \;\blacksquare$$

5.18 Theorem Let $A = \varinjlim (A_k, \alpha_{k,k+1})$, where each A_k is a finite direct sum of (simple) C^*-algebras in $\boldsymbol{\mathcal{C}}$. Assume that A is simple. Then $A \in \boldsymbol{\mathcal{C}}$.

Proof: We first note that A is purely infinite by Lemma 5.3 (2).

We now reduce to the unital case. If A is not unital, choose a nonzero projection $p \in A$ such that $p = \alpha_{l,\infty}(q)$ for some l and some projection $q \in A_l$. Then $pAp \otimes \mathcal{K} \cong A$, as in the proof of Corollary 5.16. So it suffices to show that $pAp \in \boldsymbol{\mathcal{C}}$. Now $pAp \cong \varinjlim_{k \geq l} \alpha_{l,k}(q) A \alpha_{l,k}(q)$, and Lemma 5.15 implies that each algebra in this direct system is a finite direct sum of algebras in $\boldsymbol{\mathcal{C}}$. Thus, we may assume that A is unital. Therefore we may assume that all the maps $\alpha_{k,k+1}$ are unital too.

For each k, write $A_k = \bigoplus_{j=1}^{r(k)} A_k^{(j)}$, with each $A_k^{(j)} \in \boldsymbol{\mathcal{C}}$. (To keep the notation in this proof straight, we will write indices associated with direct sums as superscripts.) Note that each $\alpha_{k,\infty}|_{A_k^{(j)}}$ is either injective or zero, and that we can drop all $A_k^{(j)}$ for which this map is zero. Thus, without loss of generality, we can assume that each $\alpha_{k,\infty}$ is injective; then so is each $\alpha_{k,k+1}$. Let $\pi_k^{(j)} : A_k \to A_k^{(j)}$ be the projection map. Now choose finite sets $F_k \subset A_k$ such that $F_k = \bigcup_{j=1}^{r(k)} F_k^{(j)}$ with $F_k^{(j)} \subset A_k^{(j)}$, such that $\alpha_{k,k+1}(F_k) \subset F_{k+1}$ for each k, and such that $\bigcup_{k=1}^{\infty} \alpha_{k,\infty}(F_k)$ is dense in the unit ball of A.

We will construct even Cuntz-circle algebras $D_k = \bigoplus_{j=1}^{r(k)} D_k^{(j)}$, finite generating subsets $H_k^{(j)}$ contained in the unit ball of $D_k^{(j)}$ and with $1_{D_k^{(j)}} \in H_k^{(j)}$, and homomorphisms $\rho_k = \bigoplus_{j=1}^{r(k)} \rho_k^{(j)} : D_k \to A_k$ and $\delta_{k,k+1} : D_k \to D_{k+1}$, as in the following approximately commutative diagram:

$$\begin{tikzcd}
D_1 \arrow[r, "\delta_{1,2}"] \arrow[d, "\rho_1"'] & D_2 \arrow[r, "\delta_{2,3}"] \arrow[d, "\rho_2"'] & D_3 \arrow[r, "\delta_{3,4}"] \arrow[d, "\rho_3"'] & \cdots \\
A_1 \arrow[r, "\alpha_{1,2}"] & A_2 \arrow[r, "\alpha_{2,3}"] & A_3 \arrow[r, "\alpha_{3,4}"] & \cdots
\end{tikzcd}$$

We will require:

(1) The k-th square approximately commutes up to $1/2^k$, that is, $\rho_{k+1} \circ \delta_{k,k+1} \overset{1/2^k}{\approx} \alpha_{k,k+1} \circ \rho_k$ with respect to $H_k = \bigcup_{j=1}^{r(k)} H_k^{(j)}$.

(2) The squares commute in KK-theory, that is,

$$[\rho_{k+1} \circ \delta_{k,k+1}] = [\alpha_{k,k+1} \circ \rho_k] \quad \text{in } KK^0(D_k, A_{k+1}).$$

(3) Each $\rho_k^{(j)}$ is injective and approximately absorbing.

(4) $\delta_{k,k+1}$ is permanently approximately absorbing.

(5) $\delta_{k,k+1}(H_k) \subset H_{k+1}$ for all k.

(6) Every point of F_k is within $1/2^{k-1}$ of some point of $\rho_k(H_k)$.

Suppose the diagram has been constructed. Let $D = \varinjlim(D_k, \delta_{k,k+1})$. Lemma 1 of [Thn] gives a homomorphism $\rho : D \to A$ such that for every k and every $a \in D_k$, we have

$$(\rho \circ \delta_{k,\infty})(a) = \lim_{l\to\infty} (\alpha_{l,\infty} \circ \rho_l \circ \delta_{k,l})(a).$$

Since each $\alpha_{l,\infty} \circ \rho_l \circ \delta_{k,l}$ is injective, this implies ρ is isometric, hence injective. One furthermore checks that for $a \in H_k$, we have

$$\|(\rho \circ \delta_{k,\infty})(a) - (\alpha_{k,\infty} \circ \rho_k)(a)\| \le \sum_{l=k}^{\infty} 1/2^l = 1/2^{k-1}.$$

It follows from (6) that every point of $\alpha_{k,\infty}(F_k)$ is within $1/2^{k-2}$ of a point of $\rho(D)$. Since $\alpha_{k,k+1}(F_k) \subset F_{k+1}$, and the union of the images of these sets in A in dense in the unit ball of A, we conclude that ρ is surjective. Thus $D \cong A$. In particular, D is purely infinite and simple. Condition (4) now implies that each $\delta_{k,\infty}$ is approximately absorbing. It follows that D, and hence A, is in $\boldsymbol{\mathcal{C}}$.

We construct the squares in the diagram one at a time, using induction. We start with the construction of ρ_1. Since each $A_1^{(j)}$ is a unital algebra in $\boldsymbol{\mathcal{C}}$, we can write $A_1^{(j)} = \varinjlim(C_l^{(j)}, \gamma_{l,l+1}^{(j)})$, where each $C_l^{(j)}$ is an even Cuntz-circle algebra and $\gamma_{l,\infty}^{(j)}$ is approximately absorbing. Choose l so large that for each j there is a subset $G^{(j)}$ of the unit ball of $C_l^{(j)}$ whose image in $A_1^{(j)}$ approximates each point of $F_1^{(j)}$ to within $1/4$. Increasing the size of $G^{(j)}$, we may assume it contains the identities of the summands and generates $C_l^{(j)}$. Use Lemma 5.5 to produce Cuntz-circle algebras $C_l^{(j)\prime}$, obtained as quotients of $C_l^{(j)}$ with quotient maps $\kappa^{(j)}$, and unital homomorphisms $\gamma_{l,\infty}^{(j)\prime} : C_l^{(j)\prime} \to A_1^{(j)}$ which are $1/8$-approximately injective with respect to the image $G^{(j)\prime}$ of $G^{(j)}$ in $C_l^{(j)\prime}$, which are approximately absorbing, and such that $\gamma_{l,\infty}^{(j)\prime} = \gamma_{l,\infty}^{(j)\prime} \circ \kappa^{(j)}$. Using Lemma 3.9, find injective approximately absorbing unital homomorphisms $\gamma_{l,\infty}^{(j)\prime\prime} : C_l^{(j)\prime} \to A_1^{(j)}$ such that $\gamma_{l,\infty}^{(j)\prime\prime} \overset{1/2}{\approx} \gamma_{l,\infty}^{(j)\prime}$ with respect to $G^{(j)\prime}$. Set $D_1^{(j)} = C_l^{(j)\prime}$, $H_1^{(j)} = G^{(j)\prime}$ and $\rho_1^{(j)} = \gamma_{l,\infty}^{(j)\prime\prime}$. Then set $D_1 = \bigoplus_{j=1}^{r(1)} D_1^{(j)}$ and $H_1 = \bigcup_{j=1}^{r(1)} H_1^{(j)}$. Note that, with these definitions, $\rho(H_1)$ generates D_1 and approximates each point of F_1 to within 1.

We now assume that D_k and ρ_k have been constructed, and we construct D_{k+1}, ρ_{k+1}, and $\delta_{k,k+1}$. Since $A_{k+1}^{(j)}$ is a unital algebra in $\boldsymbol{\mathcal{C}}$, we can write $A_{k+1}^{(j)} = \varinjlim(C_l^{(j)}, \gamma_{l,l+1}^{(j)})$, where each $C_l^{(j)}$ is an even Cuntz-circle algebra. There is l such that there are, for each j, mutually orthogonal projections $q^{(i,j)} \in C_l^{(j)}$ with $\gamma_{l,\infty}^{(j)}(q^{(i,j)})$ close enough to $(\pi_{k+1}^{(j)} \circ \alpha_{k,k+1})(1_{A_k^{(i)}})$ that there is a unitary $v \in A_{k+1}$ with $\|v - 1\| < \varepsilon_1$ and $v[\gamma_{l,\infty}^{(j)}(q^{(i,j)})]v^* = (\pi_{k+1}^{(j)} \circ \alpha_{k,k+1})(1_{A_k^{(i)}})$ for all i and j. (The number $\varepsilon_1 > 0$ will be specified later.) Using Theorem 4.1, find $l' \ge l$ and homomorphisms $\mu^{(i,j)} : D_k^{(i)} \to C_{l'}^{(j)}$ with $\mu^{(i,j)}(1) = q^{(i,j)}$ and

$$[\mu^{(i,j)}] \times [\gamma_{l',\infty}^{(j)}] = [\rho_k^{(i)}] \times [\alpha_{k,k+1}] \times [\pi_{k+1}^{(j)}] \quad \text{in } KK^0(D_k^{(i)}, A_{k+1}^{(j)}).$$

We can require that $\mu^{(i,j)}$ be permanently approximately absorbing whenever $q^{(i,j)} \neq 0$.

For $q^{(i,j)} \neq 0$, we now apply an argument similar to, but simpler than, the proof of Lemma 5.7, to the two homomorphisms $\mu^{(i,j)} : D_k^{(i)} \to q^{(i,j)} C_{l'}^{(j)} q^{(i,j)}$ and

$$v^*[(\pi_{k+1}^{(j)} \circ \alpha_{k,k+1} \circ \rho_k^{(i)})(-)]v : D_k^{(i)} \to [\gamma_{l',\infty}^{(j)}(q^{(i,j)})] A_{k+1}^{(j)} [\gamma_{l',\infty}^{(j)}(q^{(i,j)})],$$

noting that

$$[\gamma_{l',\infty}^{(j)}(q^{(i,j)})] A_{k+1}^{(j)} [\gamma_{l',\infty}^{(j)}(q^{(i,j)})] = \varinjlim_m [\gamma_{l',m}^{(j)}(q^{(i,j)})] C_{l'}^{(j)} [\gamma_{l',m}^{(j)}(q^{(i,j)})]$$

is a direct limit of even Cuntz-circle algebras. We obtain a number $l'' \geq l'$ (obtained as the maximum over i and j of suitable numbers depending on i and j) and permanently approximately absorbing homomorphisms

$$\mu^{(i,j)\prime} : D_k^{(i)} \to [\gamma_{l',l''}^{(j)}(q^{(i,j)})] C_{l''}^{(j)} [\gamma_{l',l''}^{(j)}(q^{(i,j)})]$$

such that

$$\gamma_{l'',\infty}^{(j)} \circ \mu^{(i,j)\prime} \overset{\varepsilon_2}{\approx} v^* [(\pi_{k+1}^{(j)} \circ \alpha_{k,k+1} \circ \rho_k^{(i)})(-)] v$$

with respect to $H_k^{(i)}$ for all i and j. (The number $\varepsilon_2 > 0$ will be specified below.)

The rest of the induction step is essentially the same as the construction of ρ_1 in the initial step. Choose $l''' \geq l''$ and finite generating subsets $G^{(j)}$ of the unit ball of $C_{l'''}^{(j)}$ containing the identities of the summands and $\bigcup_i (\gamma_{l'',l'''}^{(j)} \circ \mu^{(i,j)\prime})(H_k^{(i)})$, and whose image in $A_{k+1}^{(j)}$ approximates every element of $F_{k+1}^{(j)}$ to within ε_3. Use Lemma 5.5 to produce suitable quotients $D_{k+1}^{(j)}$ of $C_{l'''}^{(j)}$, with quotient maps $\kappa^{(j)}$ and ε_4-approximately injective approximately absorbing homomorphisms $\gamma_{l''',\infty}^{(j)\prime}$ to $A_{k+1}^{(j)}$. The approximate injectivity is with respect to the image $H_{k+1}^{(j)}$ of $G^{(j)}$. Then use Lemma 3.9 to replace these homomorphisms by injective approximately absorbing homomorphisms $\gamma_{l''',\infty}^{(j)\prime\prime}$ which agree to within $2\varepsilon_4$ on $H_{k+1}^{(j)}$. Define $\rho_{k+1}^{(j)}(a) = v\gamma_{l''',\infty}^{(j)\prime\prime}(a)v^*$ and

$$\delta_{k,k+1} = \bigoplus_j \sum_i \kappa^{(j)} \circ \gamma_{l'',l'''}^{(j)} \circ \mu^{(i,j)\prime}.$$

Then the square in the diagram commutes to within $2\varepsilon_1 + \varepsilon_2 + 2\varepsilon_4$ on H_k, and the image of H_{k+1} approximates every element of F_k to within $2\varepsilon_1 + \varepsilon_3 + 2\varepsilon_4$. So we choose the ε_m such that these numbers are both less than $1/2^k$. ∎

5.19 Corollary Let $A \in \mathcal{C}$ and let B be a (separable) AF algebra. Then $A \otimes B \in \mathcal{C}$.

Proof: It is trivial that $A \otimes M_n \in \mathcal{C}$ for any n. Apply Theorem 5.18. ∎

We now use Theorem 5.18 to give a classification theorem for a class of direct limits in which no restrictions are imposed on the maps of the direct systems. The building blocks will be matrix algebras over even Cuntz algebras, and the even algebras from the following definition, which do for K_1 what the the Cuntz algebras do for K_0.

5.20 Definition Let D be the Bunce-Deddens algebra [BD] whose ordered K_0-group is $\mathbf{Q}$. (See, for example, 10.11.4 of [Bl2].) For $2 \leq m < \infty$, we define the *co-Cuntz algebra* $\mathcal{Q}_m$ to be $D \otimes \mathcal{O}_m$.

5.21 Lemma We have

$$K_0(\mathcal{Q}_m) = 0 \quad \text{and} \quad K_1(\mathcal{Q}_m) \cong \mathbf{Z}/(m-1)\mathbf{Z}.$$

If m is even, then $\mathcal{Q}_m \in \mathcal{C}$. Moreover, if A is any unital C^*-algebra in $\mathcal{C}$ with $K_0(A) = 0$ and $K_1(A) \cong \mathbf{Z}/(m-1)\mathbf{Z}$, then $A \cong \mathcal{Q}_m$.

Proof: Let D be as in Definition 5.20. Note that $K_1(D) \cong \mathbf{Z}$ (see 10.11.4 of [Bl2]), and that if G is any torsion group, then $\mathbf{Q} \otimes G = \mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Q}, G) = 0$. The computation of $K_*(\mathcal{Q}_m)$ now follows from the Künneth formula [Sch1].

It is well known (Theorem 2 of [BD]) that D is a simple direct limit of C^*-algebras of the form $C(S^1) \otimes M_n$. If m is even, then $\mathcal{Q}_m \in \mathcal{C}$ as in the proof of Corollary 5.9. The last sentence follows from Theorem 5.4. ∎

5.22 Notation Let $\mathcal{C}_0$ denote the class of all simple direct limits $A = \varinjlim (A_k, \varphi_{k,k+1})$ in which each A_k is a finite direct sum of finite matrix algebras over even Cuntz algebras and even co-Cuntz algebras.

Note that no conditions are imposed on the maps in the system.

5.23 Proposition $\mathcal{C}_0 \subset \mathcal{C}$.

Proof: This is immediate from Theorem 5.18 and Lemma 5.21. ∎

We therefore get the following classification theorem for $\mathcal{C}_0$.

5.24 Theorem (1) Let $A, B \in \mathcal{C}_0$ be unital, and suppose that

$$(K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B)).$$

Then $A \cong B$.

(2) Let $A, B \in \mathcal{C}_0$ be nonunital, and suppose that

$$(K_0(A), K_1(A)) \cong (K_0(B), K_1(B)).$$

Then $A \cong B$. ∎

It is clear that for every algebra $A \in \mathcal{C}$, the groups $K_0(A)$ and $K_1(A)$ are countable torsion groups in which every element has odd order. We now prove a converse. Our result will also show that $\mathcal{C}_0 = \mathcal{C}$. We need a lemma first.

5.25 Lemma Let

$$B = \bigoplus_{i=1}^{m} B_i \quad \text{and} \quad C = \bigoplus_{j=1}^{n} C_j$$

be finite direct sums of matrix algebras over even Cuntz algebras and even co-Cuntz algebras. Let $\lambda : B \to C$ be a unital homomorphism. Then there is a unital homomorphism $\varphi : B \to C$ such that $\varphi_* = \lambda_*$ as maps from $K_*(B)$ to $K_*(C)$, and such that every partial map $\varphi_{i,j} : B_i \to C_j$ is nonzero. (The maps $\varphi_{i,j}$ are defined to be $\kappa_j \circ \varphi \circ \mu_i$, where $\kappa_j : C \to C_j$ is the quotient map and $\mu_i : B_i \to B$ is the inclusion.)

Proof: It suffices to prove this when $n = 1$, that is, when C is simple. Let $v \in C$ satisfy $v^*v = 1$ and $q = vv^* < 1$. Then $[1 - q] = 0$ in $K_0(C)$. Since C is purely infinite simple, there are nonzero mutually orthogonal projections $p_1, \dots, p_m \in C$ with $\sum_i p_i = 1 - q$ and each $[p_i] = 0$ in $K_0(C)$. It suffices to construct unital homomorphisms $\varphi_i : B_i \to p_i C p_i$ with $[\varphi_i] = 0$ in $KK^0(B_i, C)$. We will then define, for $a = (a_1, \dots, a_m) \in B$,

$$\varphi(a) = v\lambda(a)v^* + \sum_{i=1}^{m} \psi_i(a_i).$$

If C is a matrix algebra over an even Cuntz algebra, then so is $p_i C p_i$ (by Lemma 1.11). If C is a matrix algebra over an even co-Cuntz algebra, then $p_i C p_i$ is a co-Cuntz algebra (since it is in $\mathcal{C}$ and has the right K-theory). Also, B_i is given as a matrix algebra over something, but an argument as in the beginning of the proof of Theorem 4.13 enables us to assume that the matrix size is 1. So we need to prove that if B is an even Cuntz algebra or co-Cuntz algebra, and $C = M_k(C_0)$ where C_0 is an even Cuntz algebra or co-Cuntz algebra, with $[1_C] = 0$ in $K_0(C)$, then there is an (injective) unital homomorphism φ from B to C. Moreover, if C_0 is a co-Cuntz algebra, we can vary k at will.

If both B and C_0 are even Cuntz algebras, use Lemma 1.15. If both are co-Cuntz algebras, let D be as in Definition 5.20, take $C = D \otimes M_{n-1}(\mathcal{O}_n)$, and take φ to be the tensor product of id_D with a suitable map $\mathcal{O}_m \to M_{n-1}(\mathcal{O}_n)$ from Lemma 1.15. Now suppose $B = \mathcal{Q}_m$ and $C_0 = \mathcal{O}_n$. Choose $\alpha : \mathcal{O}_m \to \mathcal{O}_2$ and

$\gamma : \mathcal{O}_2 \to M_k(\mathcal{O}_n)$ as in Lemma 1.15, and use Corollary 5.13 to choose an isomorphism $\beta : D \otimes \mathcal{O}_2 \to \mathcal{O}_2$. Then let φ be the composite

$$\mathcal{Q}_m \xrightarrow{\mathrm{id} \otimes \alpha} D \otimes \mathcal{O}_2 \xrightarrow{\beta} \mathcal{O}_2 \xrightarrow{\gamma} M_k(\mathcal{O}_n).$$

Finally, suppose $B = \mathcal{O}_m$ and $C = M_k(\mathcal{Q}_n)$. We may assume $k = n - 1$. Let α, β, and γ be as above, and set $\varphi = (\mathrm{id}_D \otimes \gamma) \circ \beta^{-1} \circ \alpha$. ∎

5.26 Theorem Let G_0 and G_1 be countable abelian torsion groups in which element has odd order, and let $g \in G_0$. Then:

(1) There is a unital algebra $A \in \boldsymbol{\mathcal{C}}_0$ such that

$$(K_0(A), [1_A], K_1(A)) \cong (G_0, g, G_1).$$

(2) There is a nonunital algebra $A \in \boldsymbol{\mathcal{C}}_0$ such that

$$(K_0(A), K_1(A)) \cong (G_0, G_1).$$

Proof: Part (2) follows from part (1) by tensoring with the compact operators. (Note that the proof of Lemma 5.14 shows equally well that $\boldsymbol{\mathcal{C}}_0$ is closed under tensoring with $\mathcal{K}$.) So it suffices to prove (1).

Theorem 2.6 of [Rr1] gives simple direct limits $B^{(i)} \cong \varinjlim (B_k^{(i)}, \psi_{k,k+1}^{(i)})$, in which each $\psi_{k,k+1}^{(i)}$ is unital and each $B_k^{(i)}$ is a finite direct sum of even Cuntz algebras, such that $K_0(B^{(i)}) \cong G_i$ and $[1_{B^{(0)}}] = g$. Let D be as in Definition 5.20. Then define $A_k = B_k^{(0)} \oplus (D \otimes B_k^{(1)})$, which is a direct sum of matrix algebras over even Cuntz algebras and even co-Cuntz algebras. Use Lemma 5.25 to find a unital homomorphism $\varphi_{k,k+1} : A_k \to A_{k+1}$ which does the same thing on K-theory as $\psi_{k,k+1}^{(0)} \oplus (\mathrm{id}_D \otimes \psi_{k,k+1}^{(1)})$, and such that all the partial maps between simple summands are nonzero. Set $A = \varinjlim (A_k, \varphi_{k,k+1})$. Then A is easily seen to have the right K-theory and $[1] = g$ in $K_0(A) \cong G_0$. It is easy to check, using the condition on $\varphi_{k,k+1}$, that the algebraic direct limit of the A_k is simple. It follows from a standard argument that A is simple. So $A \in \boldsymbol{\mathcal{C}}_0$. ∎

5.27 Corollary $\boldsymbol{\mathcal{C}}_0 = \boldsymbol{\mathcal{C}}$.

Proof: It follows from Theorem 5.26 that every possible value of our invariant for $\boldsymbol{\mathcal{C}}$ is already attained for some algebra in $\boldsymbol{\mathcal{C}}_0$. ∎

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